Riesz transform associated with the fractional Fourier transform and applications in image edge detection⋆

Zunwei Fu[a], Loukas Grafakos[b], Yan Lin[c], Yue Wu[a], Shuhui Yang[c]

[a]*School of Mathematics and Statistics, Linyi University, Linyi 276000, China*
[b]*Department of Mathematics, University of Missouri, Columbia MO 65211, USA*
[c]*School of Science, China University of Mining and Technology, Beijing 100083, China*

Abstract

The fractional Hilbert transform was introduced by Zayed [30, Zayed, 1998] and has been widely used in signal processing. In view of its connection with the fractional Fourier transform, Chen, the first, second and fourth authors of this paper in [6, Chen et al., 2021] studied the fractional Hilbert transform and other fractional multiplier operators on the real line. The present paper is concerned with a natural extension of the fractional Hilbert transform to higher dimensions: this extension is the fractional Riesz transform and is given by multiplication which a suitable chirp function on the fractional Fourier transform side. In addition to a thorough study of the fractional Riesz transform, in this work we also investigate the boundedness of singular integral operators with chirp functions on rotation invariant spaces, chirp Hardy spaces and their relation to chirp BMO spaces, as well as applications of the theory of fractional multipliers in partial differential equations. Through numerical simulation, we provide physical and geometric interpretations of high-dimensional fractional multipliers. Finally, we present an application of the fractional Riesz transforms in edge detection which verifies a hypothesis insinuated in [26, Xu et al., 2016]. In fact our numerical implementation confirms that amplitude, phase, and direction information can be simultaneously extracted by controlling the order of the fractional Riesz transform.

Keywords: fractional Fourier transform, fractional Riesz transform, edge detection, chirp Hardy space, fractional multiplier

⋆This work was partially supported by the National Natural Science Foundation of China (Nos. 12071197, 11701251 and 12071052), the Natural Science Foundation of Shandong Province (Nos. ZR2019YQ04), a Simons Foundation Fellows Award (No. 819503) and a Simons Foundation Grant (No. 624733).

Email addresses: zwfu@mail.bnu.edu.cn (Zunwei Fu), grafakosl@missouri.edu (Loukas Grafakos), linyan@cumtb.edu.cn (Yan Lin), wuyue@lyu.edu.cn (Yue Wu), yang_shu_hui@163.com (Shuhui Yang)

Contents

1. Introduction

One of the fundamental operators in Fourier analysis theory is the Hilbert transform

$$H(f)(x) = \frac{1}{\pi} \text{ p.v.} \int_{\mathbb{R}} \frac{f(y)}{x-y} dy,$$

which is a continuous analogue of the conjugate Fourier series. The early studies of the Hilbert transform were based on complex analysis methods but around the 1920s these were complemented and enriched by real analysis techniques. The Hilbert transform, being the prototype of singular integrals, provided significant inspiration for the subsequent development of this subject. The work of Calderón and Zygmund [3] in 1952 furnished extensions of singular integrals to $\mathbb{R}^n$. This theory has left a big impact in analysis in view of its many applications, especially in the field of partial differential equations. Nowadays, singular integral operators are important tools in harmonic analysis but also find many applications in applied mathematics. For instance, the Hilbert transform plays a fundamental role in communication systems and digital signal processing systems, such as in filter, edge detection and modulation theory [12, 13]. As the Hilbert transform is given by convolution with the kernel $1/(\pi t)$ on the real line, in signal processing it can be understood as the output of a linear time invariant system with an impulse response of $1/(\pi t)$.

The Fourier transform is a powerful tool in the analysis and processing of stationary signals.

Definition 1.1. *We define the Fourier transform of a function f in the Schwartz class $S(\mathbb{R}^n)$ by*

$$\hat{f}(\boldsymbol{\xi}) = \mathcal{F}(f)(\boldsymbol{\xi}) = \frac{1}{\left(\sqrt{2\pi}\right)^n} \int_{\mathbb{R}^n} f(\boldsymbol{x}) e^{-i\boldsymbol{x}\cdot\boldsymbol{\xi}} d\boldsymbol{x}.$$

In time-frequency analysis, the Hilbert transform is also known as a $\pi/2$-phase shifter. The Hilbert transform can be defined in terms of the Fourier transform as the following multiplier operator

$$\mathcal{F}(Hf)(x) = -i\mathrm{sgn}(x)\mathcal{F}(f)(x). \tag{1.1}$$

It can be seen from (1.1) that the Hilbert transform is a phase-shift converter that multiplies the positive frequency portion of the original signal by $-i$; in other words, it maintains the same amplitude and shifts the phase by $-\pi/2$, while the negative frequency portion is shifted by $\pi/2$.

The Riesz transform is a generalization of the Hilbert transform in the n-dimensional case and is also a singular integral operator, with properties analogous to those of the Hilbert transform on $\mathbb{R}$. It is defined as

$$R_j(f)(\boldsymbol{x}) = c_n \,\text{p.v.} \int_{\mathbb{R}^n} \frac{x_j - y_j}{|\boldsymbol{x} - \boldsymbol{y}|^{n+1}} f(\boldsymbol{y}) d\boldsymbol{y}, \quad 1 \le j \le n,$$

where $c_n = \Gamma(\frac{n+1}{2})/\pi^{\frac{n+1}{2}}$. The Riesz transform is also a multiplier operator

$$\mathcal{F}(R_j f)(\boldsymbol{x}) = -\frac{ix_j}{|\boldsymbol{x}|}\mathcal{F}(f)(\boldsymbol{x}).$$

Remark 1.1. The multiplier of the Hilbert transform is $-i\mathrm{sgn}(x)$, and it is simply a phase-shift converter. The multiplier of the Riesz transform is $-ix_j/|\boldsymbol{x}|$, and thus, the Riesz transform is not only a phase-shift converter but also an amplitude attenuator.

The Riesz transform has wide applications in image edge detection, image quality assessment and biometric feature recognition [16, 33, 34].

The Fourier transform is limited in processing and analyzing nonstationary signals. The fractional Fourier transform (FRFT) was proposed and developed by some scholars mainly because of the need for nonstationary signals. The FRFT originated in the work of Wiener in [29]. Namias in [20] proposed the FRFT through a method that was primarily based on eigenfunction expansions in 1980. McBride-Kerr in [19] and Kerr in [14] provided integral expressions of the FRFT on $S(\mathbb{R})$ and $L^2(\mathbb{R})$, respectively. In [6], Chen, and the first, second and fourth authors of this paper, established the behavior of FRFT on $L^p(\mathbb{R})$ for $1 \le p < 2$.

A chirp function is a nonstationary signal in which the frequency increases (upchirp) or decreases (downchirp) with time. The chirp signal is the most common nonstationary signal. In 1998, Zayed in [30] gave the following definition of the fractional

Hilbert transform

$$H_\alpha(f)(x) = \frac{1}{\pi}\,\text{p.v. } e_{-\alpha}(x)\int_{\mathbb{R}} \frac{f(y)}{x-y} e_\alpha(y)dy,$$

where $e_\alpha(x) = e^{\frac{ix^2\cot\alpha}{2}}$ is a chirp function.

In [22], Pei and Yeh expressed the discrete fractional Hilbert transform as a composition of the discrete fractional Fourier transform (DFRFT), a multiplier, and the inverse DFRFT; based on this they conducted simulation verification on the edge detection of digital images. In [6], Chen, and the first, second and fourth authors of this paper related the fractional Hilbert transform to the fractional Fourier multiplier

$$\mathcal{F}_\alpha(H_\alpha f)(x) = -i\text{sgn}((\pi-\alpha)x)\mathcal{F}_\alpha(f)(x),$$

where $\mathcal{F}_\alpha$ is FRFT; see Definition 1.2. In analogy with the Hilbert transform, the fractional Hilbert transform is also a phase-shift converter. As indicated above, the continuous fractional Hilbert transform can be decomposed into a composition of the FRFT, a multiplier, and the inverse FRFT. The fractional Hilbert transform can also be used in single sideband communication systems and image encryption systems. The rotation angle can be used as the encryption key to improve the communication security and image encryption effect in [28].

The multidimensional FRFT has recently made its appearance: Zayed [31, 32] introduced a new two-dimensional FRFT. In [15], Kamalakkannan and Roopkumar introduced the multidimensional FRFT.

Definition 1.2. ([15]) *Let α_k be real numbers. The multidimensional FRFT with order $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ on $L^1(\mathbb{R}^n)$ is defined by*

$$\mathcal{F}_{\boldsymbol{\alpha}}(f)(\boldsymbol{u}) = \int_{\mathbb{R}^n} f(\boldsymbol{x})K_{\boldsymbol{\alpha}}(\boldsymbol{x}, \boldsymbol{u})d\boldsymbol{x},$$

where $K_{\boldsymbol{\alpha}}(\boldsymbol{x}, \boldsymbol{u}) = \prod_{k=1}^n K_{\alpha_k}(x_k, u_k)$ and $K_{\alpha_k}(x_k, u_k)$ are given by

$$K_{\alpha_k}(x_k, u_k) = \begin{cases} \frac{c(\alpha_k)}{\sqrt{2\pi}} e^{i(a(\alpha_k)(x_k^2+u_k^2-2b(\alpha_k)x_k u_k))}, & \alpha_k \notin \pi\mathbb{Z}, \\ \delta(x_k - u_k), & \alpha_k \in 2\pi\mathbb{Z}, \\ \delta(x_k + u_k), & \alpha_k \in 2\pi\mathbb{Z} + \pi, \end{cases}$$

$\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$, $a(\alpha_k) = \frac{\cot(\alpha_k)}{2}$, $b(\alpha_k) = \sec(\alpha_k)$, $c(\alpha_k) = \sqrt{1 - i\cot(\alpha_k)}$.

Remark 1.2. Suppose that $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n$ with $\alpha_k \notin \pi\mathbb{Z}$ for all $k = 1, 2, \ldots, n$. Consider the chirps

$$e_{\boldsymbol{\alpha}}(\boldsymbol{x}) = e^{i\sum_{k=1}^n a(\alpha_k)x_k^2},$$

for $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$. It is straightforward that the FRFT of f can be written as

$$\mathcal{F}_\alpha(f)(\boldsymbol{u}) = c(\boldsymbol{\alpha})e_\alpha(\boldsymbol{u})\mathcal{F}(e_\alpha f)(\tilde{\boldsymbol{u}}),$$

where $c(\boldsymbol{\alpha}) = c(\alpha_1)\cdots c(\alpha_n)$, $\tilde{\boldsymbol{u}} = (u_1 \csc\alpha_1, \ldots, u_n \csc\alpha_n)$. From the preceding identity, it can be seen that $\mathcal{F}_\alpha$ is bounded from $S(\mathbb{R}^n)$ to $S(\mathbb{R}^n)$. We rewrite

$$K_\alpha(\boldsymbol{x}, \boldsymbol{u}) = \frac{c(\boldsymbol{\alpha})}{\left(\sqrt{2\pi}\right)^n} e_\alpha(\boldsymbol{x})e_\alpha(\boldsymbol{u})e^{-i\sum_{k=1}^n x_k u_k \csc\alpha_k}.$$

We now define the fractional Riesz transform associated with the multidimensional FRFT as follows:

Definition 1.3. *For $1 \le j \le n$, the jth fractional Riesz transform R_j^α is defined on the FRFT side by multiplication by the function $-i\frac{\tilde{u}_j}{|\tilde{\boldsymbol{u}}|}$. That is, for any $f \in S(\mathbb{R}^n)$,*

$$\mathcal{F}_\alpha\left(R_j^\alpha f\right)(\boldsymbol{u}) = -i\frac{\tilde{u}_j}{|\tilde{\boldsymbol{u}}|}\mathcal{F}_\alpha(f)(\boldsymbol{u}),$$

where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ with $\alpha_k \notin \pi\mathbb{Z}$, $k = 1, 2, \ldots, n$; $\boldsymbol{u} = (u_1, \ldots, u_n)$ and $\tilde{\boldsymbol{u}} = (u_1 \csc\alpha_1, \ldots, u_n \csc\alpha_n) = (\tilde{u}_1, \ldots, \tilde{u}_n)$.

This paper is organized as follows. In Section 2, we obtain characterizations of the fractional Riesz transform in terms of the FRFT and we note that the fractional Riesz transform is not only a phase shift converter but also an amplitude attenuator. We identify behavior of the fractional Riesz transform in terms of dilations, translations, modulations, and chirp multiplications. We obtain the identity $\sum_{j=1}^n (R_j^\alpha)^2 = -I$ and the boundedness of singular integral operators with a chirp function on rotation invariant spaces. In Section 3, we derive a formula for the high-dimensional FRFT and we provide an application of the fractional Riesz transform to partial differential equations. In Section 4, we conduct a simulation experiment with the fractional Riesz transform on an image and give the physical and geometric interpretation of the high-dimensional fractional multiplier theorem. In Section 5, we discuss a situation where it is difficult to directly use the fractional Riesz transform for edge detection but the fractional multiplier theorem provides this possibility. The use of the fractional Riesz transform is completely equivalent to the compound operation of the FRFT, inverse FRFT and multiplier, and the FRFT and inverse FRFT can realize fast operations. In Section 6, we introduce the definition of the chirp Hardy space by taking the Possion maximum for the function with the chirp factor and study the dual spaces of chirp Hardy spaces. We also characterize the boundedness of singular integral operators with chirp functions on chirp Hardy spaces.

2. Fractional Riesz transforms

2.1. Properties of the fractional Riesz transforms

Theorem 2.1. *For $1 \le j \le n$, the jth fractional Riesz transform of $f \in S(\mathbb{R}^n)$ is given by*

$$R_j^\alpha(f)(\boldsymbol{x}) = c_n \text{ p.v. } e_{-\alpha}(\boldsymbol{x}) \int_{\mathbb{R}^n} \frac{x_j - y_j}{|\boldsymbol{x}-\boldsymbol{y}|^{n+1}} f(\boldsymbol{y}) e_\alpha(\boldsymbol{y}) d\boldsymbol{y}, \qquad \boldsymbol{x} \in \mathbb{R}^n,$$

where $c_n = \Gamma(\frac{n+1}{2})/\pi^{\frac{n+1}{2}}$ and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ with $\alpha_k \notin \pi\mathbb{Z}$, $k = 1, 2, \ldots, n$.

Proof. Fix a $f \in S(\mathbb{R}^n)$. For $1 \le j \le n$, we have

$$\begin{aligned}
\mathcal{F}_\alpha\left(R_j^\alpha f\right)(\boldsymbol{u}) &= \int_{\mathbb{R}^n} R_j^\alpha f(\boldsymbol{x}) \frac{c(\boldsymbol{\alpha})}{\left(\sqrt{2\pi}\right)^n} e_\alpha(\boldsymbol{x}) e_\alpha(\boldsymbol{u}) e^{-i\sum_{j=1}^n x_j u_j \csc\alpha_j} d\boldsymbol{x} \\
&= \int_{\mathbb{R}^n} \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} e_{-\alpha}(\boldsymbol{x}) \lim_{\varepsilon\to 0} \int_{|\boldsymbol{y}|\ge\varepsilon} \frac{y_j}{|\boldsymbol{y}|^{n+1}} f(\boldsymbol{x}-\boldsymbol{y}) e_\alpha(\boldsymbol{x}-\boldsymbol{y}) d\boldsymbol{y} \\
&\qquad \times \frac{c(\boldsymbol{\alpha})}{\left(\sqrt{2\pi}\right)^n} e_\alpha(\boldsymbol{x}) e_\alpha(\boldsymbol{u}) e^{-i\sum_{j=1}^n x_j u_j \csc\alpha_j} d\boldsymbol{x} \\
&= \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \lim_{\varepsilon\to 0} \int_{|\boldsymbol{y}|\ge\varepsilon} \frac{y_j}{|\boldsymbol{y}|^{n+1}} \int_{\mathbb{R}^n} f(\boldsymbol{x}-\boldsymbol{y}) e_\alpha(\boldsymbol{x}-\boldsymbol{y}) \frac{c(\boldsymbol{\alpha})}{\left(\sqrt{2\pi}\right)^n} e_\alpha(\boldsymbol{u}) \\
&\qquad \times e^{-i\sum_{j=1}^n x_j u_j \csc\alpha_j} d\boldsymbol{x} d\boldsymbol{y}.
\end{aligned}$$

Changing variables yields

$$\begin{aligned}
\mathcal{F}_\alpha\left(R_j^\alpha f\right)(\boldsymbol{u}) &= \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \lim_{\varepsilon\to 0} \int_{|\boldsymbol{y}|\ge\varepsilon} \frac{y_j}{|\boldsymbol{y}|^{n+1}} \int_{\mathbb{R}^n} f(\boldsymbol{w}) e_\alpha(\boldsymbol{w}) \frac{c(\boldsymbol{\alpha})}{\left(\sqrt{2\pi}\right)^n} e_\alpha(\boldsymbol{u}) \\
&\qquad \times e^{-i\sum_{j=1}^n (y_j+w_j) u_j \csc\alpha_j} d\boldsymbol{w} d\boldsymbol{y} \\
&= \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \lim_{\varepsilon\to 0} \int_{|\boldsymbol{y}|\ge\varepsilon} \frac{y_j}{|\boldsymbol{y}|^{n+1}} e^{-i\sum_{j=1}^n y_j u_j \csc\alpha_j} d\boldsymbol{y} \int_{\mathbb{R}^n} \frac{c(\alpha)}{\left(\sqrt{2\pi}\right)^n} \\
&\qquad \times f(\boldsymbol{w}) e_\alpha(\boldsymbol{u}) e_\alpha(\boldsymbol{w}) e^{-i\sum_{j=1}^n w_j u_j \csc\alpha_j} d\boldsymbol{w} \\
&= \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \mathcal{F}_\alpha(f)(\boldsymbol{u}) \lim_{\varepsilon\to 0} \int_{|\boldsymbol{y}|\ge\varepsilon} \frac{y_j}{|\boldsymbol{y}|^{n+1}} e^{-i\sum_{j=1}^n y_j u_j \csc\alpha_j} d\boldsymbol{y} \\
&= \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \mathcal{F}_\alpha(f)(\boldsymbol{u}) \lim_{\varepsilon\to 0} \int_{\varepsilon\le|\boldsymbol{y}|\le\frac{1}{\varepsilon}} \frac{y_j}{|\boldsymbol{y}|^{n+1}} e^{-i\boldsymbol{y}\cdot\tilde{\boldsymbol{u}}} d\boldsymbol{y}.
\end{aligned}$$

Switching to polar coordinates and using Lemma 5.1.15 in [10] we obtain

$$\mathcal{F}_\alpha\left(R_j^\alpha f\right)(\boldsymbol{u}) = \frac{-i\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \mathcal{F}_\alpha(f)(\boldsymbol{u}) \lim_{\varepsilon\to 0} \int_{S^{n-1}} \int_{\varepsilon\le r\le\frac{1}{\varepsilon}} \sin(r\tilde{\boldsymbol{u}}\cdot\boldsymbol{\theta}) \frac{r}{r^{n+1}} r^{n-1} dr \theta_j d\boldsymbol{\theta}$$

$$= \frac{-i\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \mathcal{F}_\alpha(f)(\boldsymbol{u}) \int_{s^{n-1}} \int_0^\infty \sin(r\tilde{\boldsymbol{u}} \cdot \boldsymbol{\theta}) \frac{dr}{r} \theta_j d\boldsymbol{\theta}$$
$$= \frac{-i\Gamma(\frac{n+1}{2})}{2\pi^{\frac{n-1}{2}}} \mathcal{F}_\alpha(f)(\boldsymbol{u}) \int_{s^{n-1}} \operatorname{sgn}(\tilde{\boldsymbol{u}} \cdot \boldsymbol{\theta}) \theta_j d\boldsymbol{\theta}$$
$$= -i \frac{\tilde{u}_j}{|\tilde{\boldsymbol{u}}|} \mathcal{F}_\alpha(f)(\boldsymbol{u}),$$

and this completes the proof of the theorem. □

Remark 2.1. The fractional Riesz transform reduces to the fractional Hilbert transform for $n = 1$, while the fractional Riesz transform reduces to the classical Riesz transform for $\boldsymbol{\alpha} = (\frac{\pi}{2} + k_1\pi, \frac{\pi}{2} + k_2\pi, \ldots, \frac{\pi}{2} + k_n\pi),\ k_j \in \mathbb{Z},\ j = 1, 2, \ldots, m.$

Next, we examine how the fractional Riesz transform interacts with respect to the operations of dilation, translation, modulation, and chirp multiplication.

Proposition 2.2. *Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ with $\alpha_k \notin \pi\mathbb{Z},\ k = 1, 2, \ldots, n$, and $f \in S(\mathbb{R}^n)$.*

(i) (Dilation) *Let $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n) \in \mathbb{R}^n$ with $\beta_k \notin \pi\mathbb{Z},\ k = 1, 2, \ldots, n$. For any $c > 0$, $\boldsymbol{x} \in \mathbb{R}^n$, and $c \in \mathbb{R}$, one has*

$$R_j^{\alpha}(f(c\,\cdot))(\boldsymbol{x}) = R_j^{\boldsymbol{\beta}}(f)(c\boldsymbol{x}),$$

where $a(\beta_k) = \frac{a(\alpha_k)}{c^2}$.

(ii) (Translation and modulation) *For any $\boldsymbol{h} = (h_1, \ldots, h_n) \in \mathbb{R}^n$, and all $\boldsymbol{x} \in \mathbb{R}^n$ we have*

$$R_j^{\alpha}(f(\cdot - \boldsymbol{h}))(\boldsymbol{x}) = e_\alpha(\sqrt{2}\boldsymbol{h}) e^{-i\boldsymbol{h}_\alpha \boldsymbol{x}} R_j^{\alpha}\left(f e^{i\boldsymbol{h}_\alpha \cdot}\right)(\boldsymbol{x} - \boldsymbol{h}),$$

where $\boldsymbol{h}_\alpha = (h_1 a(\alpha_1), \ldots, h_n a(\alpha_n))$.

(iii) (Chirp multiplication) *For all $\boldsymbol{x} \in \mathbb{R}^n$ the identity below is valid*

$$R_j^{\alpha}(e_{-\alpha} f)(\boldsymbol{x}) = e_{-\alpha}(\boldsymbol{x}) R_j(f)(\boldsymbol{x}).$$

Proof. (i) For any $f \in S(\mathbb{R}^n)$, by the substitution of variables, it follows that

$$R_j^{\alpha}(f(c\,\cdot))(\boldsymbol{x}) = c_n \text{ p.v. } e_{-\alpha}(\boldsymbol{x}) \int_{\mathbb{R}^n} \frac{x_j - y_j}{|\boldsymbol{x} - \boldsymbol{y}|^{n+1}} f(c\boldsymbol{y}) e_\alpha(\boldsymbol{y}) d\boldsymbol{y}$$
$$= c_n \text{ p.v. } e_{-\alpha}(\boldsymbol{x}) \int_{\mathbb{R}^n} \frac{cx_j - s_j}{|c\boldsymbol{x} - \boldsymbol{s}|^{n+1}} f(\boldsymbol{s}) e_\alpha\left(\frac{\boldsymbol{s}}{c}\right) d\boldsymbol{s}$$
$$= c_n \text{ p.v. } e^{i\Sigma_{k=1}^n a(-\alpha_k) x_k^2} \int_{\mathbb{R}^n} \frac{cx_j - s_j}{|c\boldsymbol{x} - \boldsymbol{s}|^{n+1}} f(\boldsymbol{s}) e^{i\Sigma_{k=1}^n \frac{a(\alpha_k)}{c^2} s_k^2} d\boldsymbol{s}$$
$$= c_n \text{ p.v. } e^{i\Sigma_{k=1}^n a(-\beta_k)(cx_k)^2} \int_{\mathbb{R}^n} \frac{cx_j - s_j}{|c\boldsymbol{x} - \boldsymbol{s}|^{n+1}} f(\boldsymbol{s}) e^{i\Sigma_{k=1}^n a(\beta_k) s_k^2} d\boldsymbol{s}$$

$$= R_j^{\beta}(f)(c\boldsymbol{x}),$$

where $a(\beta_k) = \frac{a(\alpha_k)}{c^2}$.

(ii) For any $f \in S(\mathbb{R}^n)$ and $\boldsymbol{h} \in \mathbb{R}^n$, by the substitution of variables, we deduce that

$$\begin{aligned}
R_j^{\alpha}(f(\cdot - \boldsymbol{h}))(\boldsymbol{x}) &= c_n \text{ p.v. } e_{-\alpha}(\boldsymbol{x}) \int_{\mathbb{R}^n} \frac{x_j - y_j}{|\boldsymbol{x}-\boldsymbol{y}|^{n+1}} f(\boldsymbol{y}-\boldsymbol{h}) e_{\alpha}(\boldsymbol{y}) d\boldsymbol{y} \\
&= c_n \text{ p.v. } e_{-\alpha}(\boldsymbol{x}) \int_{\mathbb{R}^n} \frac{x_j - h_j - s_j}{|\boldsymbol{x}-\boldsymbol{h}-\boldsymbol{s}|^{n+1}} f(\boldsymbol{s}) e_{\alpha}(\boldsymbol{s}+\boldsymbol{h}) d\boldsymbol{s} \\
&= c_n \text{ p.v. } e_{-\alpha}(\boldsymbol{x}) e_{\alpha}(\boldsymbol{h}) \int_{\mathbb{R}^n} \frac{x_j - h_j - s_j}{|\boldsymbol{x}-\boldsymbol{h}-\boldsymbol{s}|^{n+1}} f(\boldsymbol{s}) e^{i\boldsymbol{h}_{\alpha}\boldsymbol{s}} e_{\alpha}(\boldsymbol{s}) d\boldsymbol{s} \\
&= e_{\alpha}(\sqrt{2}\boldsymbol{h}) e^{-i\boldsymbol{h}_{\alpha}\boldsymbol{x}} R_j^{\alpha}\left(f e^{i\boldsymbol{h}_{\alpha}\cdot}\right)(\boldsymbol{x}-\boldsymbol{h}),
\end{aligned}$$

where $\boldsymbol{h}_{\alpha} = (h_1 a(\alpha_1), \ldots, h_n a(\alpha_n))$.

(iii) This property is easily obtained from the integral definition of the Riesz transform and the fractional Riesz transform. We omit the details.

This finishes the proof of proposition 2.2. □

Lemma 2.3. *(FRFT inversion theorem)* ([15]) *Suppose $f \in S(\mathbb{R}^n)$. Then*

$$f(\boldsymbol{x}) = \int_{\mathbb{R}^n} \mathcal{F}_{\alpha}(f)(\boldsymbol{u}) K_{-\alpha}(\boldsymbol{u}, \boldsymbol{x}) d\boldsymbol{u}, \quad a.e.\ x \in \mathbb{R}^n.$$

By Definition 1.3 and Lemma 2.3, the jth fractional Riesz transform of order $\boldsymbol{\alpha}$ can be rewritten as

$$(R_j^{\alpha} f)(\boldsymbol{x}) = \left[\mathcal{F}_{-\alpha}\left(-i\frac{\tilde{u}_j}{|\tilde{\boldsymbol{u}}|}(\mathcal{F}_{\alpha} f)(\boldsymbol{u})\right)\right](\boldsymbol{x}).$$

Denote $m_j^{\alpha}(\boldsymbol{u}) := -i\tilde{u}_j/|\tilde{\boldsymbol{u}}|$. It can be seen that the fractional Riesz transform of a function f can be decomposed into three simpler operators, according to the diagram of Fig. 2.1:

(i) FRFT of order $\boldsymbol{\alpha}$, $g(\boldsymbol{u}) = (\mathcal{F}_{\alpha} f)(\boldsymbol{u})$;
(ii) multiplication by a fractional L^p multiplier, $h(\boldsymbol{u}) = m_j^{\alpha}(\boldsymbol{u}) g(\boldsymbol{u})$;
(iii) FRFT of order $-\boldsymbol{\alpha}$, $(R_j^{\alpha} f)(\boldsymbol{x}) = (\mathcal{F}_{-\alpha} h)(\boldsymbol{x})$

Consider a 2-dimensional fractional Riesz transform as an example. It can be seen from Definition 1.3 that the fractional Riesz transform of order $\boldsymbol{\alpha}$ is a phase-shift converter that multiplies the positive portion in the $\boldsymbol{\alpha}$-order fractional Fourier domain of signal f by $-i$; in other words, it shifts the phase by $-\pi/2$ while the negative portion of $\mathcal{F}_{\alpha} f$ is shifted by $\pi/2$. It is also an amplitude reducer that multiplies the amplitude in the $\boldsymbol{\alpha}$-order fractional Fourier domain of signal f by $\tilde{u}_j/|\tilde{\boldsymbol{u}}|$, as shown in Fig. 2.2.

Next, we establish the $L^p(\mathbb{R}^n)$ boundedness of the fractional Riesz transform.

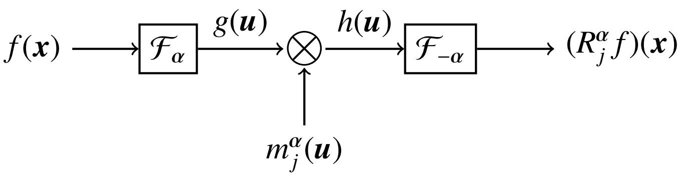


Figure 2.1: The decomposition of the jth fractional Riesz transform.

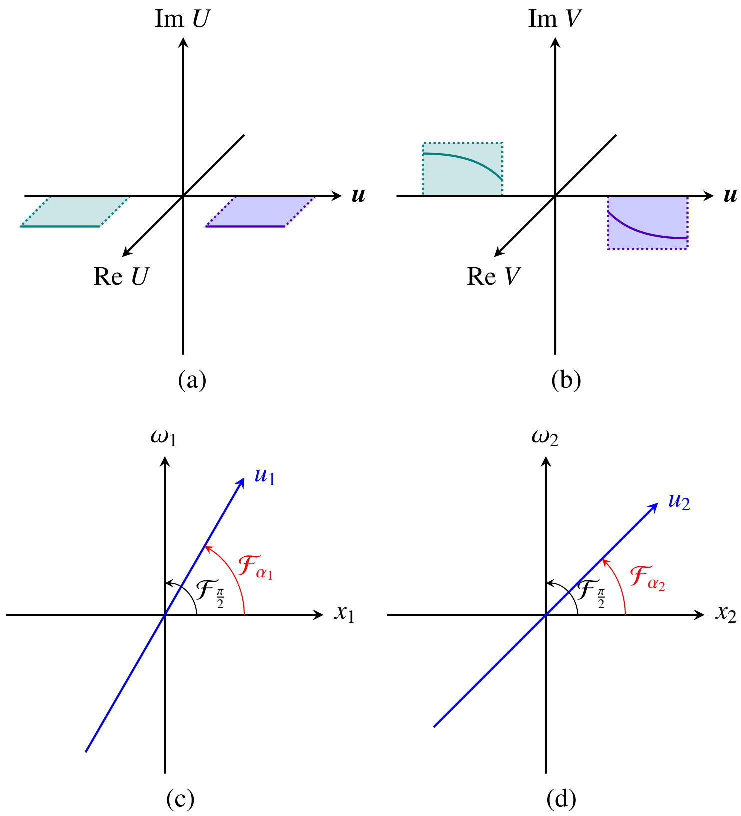


Figure 2.2: (a) the original signal: $U = (\mathcal{F}_\alpha f)(\boldsymbol{u})$; (b) after fractional Riesz transform of order $\boldsymbol{\alpha}$: $V = (\mathcal{F}_\alpha(R_j^\alpha f)(\boldsymbol{u})$; (c)-(d) rotations of the time-frequency planes, $\boldsymbol{u} = (u_1, u_2)$, $\boldsymbol{x} = (x_1, x_2)$, $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$.

Theorem 2.4. *For all $1 < p < \infty$, there exists a positive constant C such that*

$$\left\|R_j^\alpha(f)\right\|_{L^p} \leq C\,\|f\|_{L^p}\,,$$

for all f in $S(\mathbb{R}^n)$.

Proof. From the L^p boundedness of the Riesz transform in [18] with Theorem 2.1.4, it follows that

$$\begin{aligned}\left\|R_j^\alpha(f)\right\|_{L^p} &= \left(\int_{\mathbb{R}^n}\left|c_n e_{-\alpha}(\boldsymbol{x})\int_{\mathbb{R}^n}\frac{y_j}{|\boldsymbol{y}|^{n+1}}f(\boldsymbol{x}-\boldsymbol{y})e_\alpha(\boldsymbol{x}-\boldsymbol{y})d\boldsymbol{y}\right|^p d\boldsymbol{x}\right)^{\frac{1}{p}}\\ &= \left\|R_j(fe_\alpha)\right\|_{L^p}\\ &\leq C\|f\|_{L^p},\end{aligned}$$

for all f in $S(\mathbb{R}^n)$. □

As a consequence of Definition 1.3, we derive the following identity for the fractional Riesz transform.

Theorem 2.5. *The fractional Riesz transforms satisty*

$$\sum_{j=1}^{n}\left(R_j^\alpha\right)^2 = -I, \quad \textit{on } L^2(\mathbb{R}^n),$$

where I is the identity operator.

Proof. Use the FRFT and the identity $\sum_{j=1}^n\left(-i\tilde{u}_j/|\tilde{\boldsymbol{u}}|\right)^2 = -1$ to obtain

$$\begin{aligned}\mathcal{F}_\alpha\left(\sum_{j=1}^n\left(R_j^\alpha\right)^2 f\right)(\boldsymbol{u}) &= \sum_{j=1}^n\left(-i\frac{\tilde{u}_j}{|\tilde{\boldsymbol{u}}|}\right)^2\mathcal{F}_\alpha(f)(\boldsymbol{u})\\ &= -\mathcal{F}_\alpha(f)(\boldsymbol{u}),\end{aligned}$$

for any f in $L^2(\mathbb{R}^n)$. □

2.2. The boundedness of singular integral operators with chirp functions on rotation-invariant spaces

Just like the Riesz transforms, the fractional Riesz transforms are singular integral operators. In fact they are special cases of more general singular integral operators whose kernels K are equipped with chirp functions

$$T_\alpha(f)(\boldsymbol{x}) = \text{p.v.}\int_{\mathbb{R}^n}e_{-\alpha}(\boldsymbol{x})K(\boldsymbol{x},\boldsymbol{y})e_\alpha(\boldsymbol{y})f(\boldsymbol{y})d\boldsymbol{y} = \text{p.v.}\int_{\mathbb{R}^n}K^\alpha(\boldsymbol{x},\boldsymbol{y})f(\boldsymbol{y})d\boldsymbol{y}.$$

When $\boldsymbol{\alpha} = (\frac{\pi}{2}+k_1\pi, \frac{\pi}{2}+k_2\pi, \ldots, \frac{\pi}{2}+k_n\pi), k_j\in\mathbb{Z}$, $T_{\boldsymbol{\alpha}}$ can be regarded as the classical singular integral operators:

$$T(f)(\boldsymbol{x}) = \text{p.v.}\int_{\mathbb{R}^n}K(\boldsymbol{x},\boldsymbol{y})f(\boldsymbol{y})d\boldsymbol{y}.$$

Then, we consider the boundedness of $T_{\boldsymbol{\alpha}}$ on rotation invariant Banach spaces.

Definition 2.6. *Suppose that $(X, \|\cdot\|_X)$ is a Banach function space of complex-valued functions defined on $\mathbb{R}^n$. We say that X a rotation-invariant space if*

$$\|e_\alpha f\|_X = \|f\|_X,$$

for any $f \in X$, where $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ with $\alpha_k \notin \pi\mathbb{Z}$ for all $k = 1, 2, \dots, n$.

When K satisfies suitable conditions, the boundedness of T_α and T on rotation invariant space are equivalent.

Theorem 2.7. *If X is a rotation invariant space, then T is bounded from X to X if and only if T_α is bounded from X to X.*

Proof. Let $f \in X$ and T satisfy $\|T\|_{X \to X} < \infty$. We have that

$$\|T_\alpha(f)\|_X = \|T(e_\alpha f)\|_X \leq C\|e_\alpha f\|_X = C\|f\|_X.$$

Conversely, for $\|T_\alpha\|_{X \to X} < \infty$, we obtain

$$\|T(f)\|_X = \|e_{-\alpha} T(f)\|_X = \|T_\alpha(e_{-\alpha} f)\|_X \leq C\|e_{-\alpha} f\|_X = C\|f\|_X.$$

Hence, the theorem follows. □

3. Application of the fractional Riesz transform in partial differential equations

The fractional Riesz transforms can be used to reconcile various combinations of partial derivatives of functions. We first established the derivative formula of the FRFT.

Lemma 3.1. (FRFT derivative formula) *Suppose that $f \in L^1(R^n)$. If $e_\alpha f$ is absolutely continuous on $\mathbb{R}^n$ with respect to the kth variable, we have*

$$\mathcal{F}_\alpha\left(e_{-\alpha}\frac{\partial[e_\alpha f]}{\partial y_k}\right)(\boldsymbol{x}) = i x_k \csc\alpha_k \mathcal{F}_\alpha(f)(\boldsymbol{x}),$$

where $\boldsymbol{x} = (x_1, x_2, \dots, x_n)$ and $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{R}^n$ with $\alpha_k \notin \pi\mathbb{Z},\ k = 1, 2, \dots, n$.

Proof. Let $\boldsymbol{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$. Since $e_\alpha f$ is absolutely continuous on $\mathbb{R}^n$ with respect to the kth variable, we can get that $\frac{\partial[e_\alpha(\boldsymbol{y})f(\boldsymbol{y})]}{\partial y_k} \in L^1(\mathbb{R}^n)$. For $f \in L^1(\mathbb{R}^n)$, we have

$$\begin{aligned}\mathcal{F}_\alpha\left(e_{-\alpha}(\boldsymbol{y})\frac{\partial[e_\alpha(\boldsymbol{y})f(\boldsymbol{y})]}{\partial y_k}\right)(\boldsymbol{x}) &= \frac{c(\boldsymbol{\alpha})}{\left(\sqrt{2\pi}\right)^n}\int_{\mathbb{R}^n}\left(e_{-\alpha}(\boldsymbol{y})\frac{\partial[e_\alpha(\boldsymbol{y})f(\boldsymbol{y})]}{\partial y_k}\right)e_\alpha(\boldsymbol{y}) \\ &\qquad \times e_\alpha(\boldsymbol{x})e^{-i\sum_{j=1}^n x_j y_j \csc\alpha_j}d\boldsymbol{y} \\ &= \frac{c(\boldsymbol{\alpha})}{\left(\sqrt{2\pi}\right)^n}e_\alpha(\boldsymbol{x})\int_{\mathbb{R}^n}\frac{\partial[e_\alpha(\boldsymbol{y})f(\boldsymbol{y})]}{\partial y_k}e^{-i\sum_{j=1}^n x_j y_j \csc\alpha_j}d\boldsymbol{y}\end{aligned}$$

$$
= \frac{c(\boldsymbol{\alpha})e_{\boldsymbol{\alpha}}(\boldsymbol{x})}{\left(\sqrt{2\pi}\right)^n}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}}\frac{\partial[e_\alpha(\boldsymbol{y})f(\boldsymbol{y})]}{\partial y_k}e^{-ix_ky_k\csc\alpha_k}dy_k
\times \prod_{j=1,j\neq k}^{n} e^{-ix_jy_j\csc\alpha_j}\prod_{j=1,j\neq k}^{n} dy_j.
$$

As $e_\alpha f$ is absolutely continuous on $\mathbb{R}^n$ with respect to the kth variable, integration by parts yields

$$
\int_{\mathbb{R}}\frac{\partial[e_\alpha(\boldsymbol{y})f(\boldsymbol{y})]}{\partial y_k}e^{-ix_ky_k\csc\alpha_k}dy_k = ix_k\csc\alpha_k\int_{\mathbb{R}} e_\alpha(\boldsymbol{y})f(\boldsymbol{y})e^{-ix_ky_k\csc\alpha_k}dy_k.
$$

Then

$$
\begin{aligned}
\mathcal{F}_\alpha\left(e_{-\alpha}(\boldsymbol{y})\frac{\partial[e_\alpha(\boldsymbol{y})f(\boldsymbol{y})]}{\partial y_k}\right)(\boldsymbol{x}) &= ix_k\csc\alpha_k\frac{c(\boldsymbol{\alpha})}{(\sqrt{2\pi})^n}\int_{\mathbb{R}^n} e_\alpha(\boldsymbol{y})f(\boldsymbol{y})e_{\boldsymbol{\alpha}}(\boldsymbol{x})e^{-i\sum_{j=1}^n x_jy_j\csc\alpha_j}d\boldsymbol{y}\\
&= ix_k\csc\alpha_k\mathcal{F}_\alpha(f)(\boldsymbol{x}),
\end{aligned}
$$

and this completes the proof of the lemma. □

Lemma 3.2. (FRFT derivative formula) *Suppose that $f\in L^1(\mathbb{R}^n)$ and $x_kf(\boldsymbol{x})\in L^1(\mathbb{R}^n)$. Then, we have*

$$
\frac{\partial(e_{-\boldsymbol{\alpha}}(\boldsymbol{x})\mathcal{F}_\alpha(f)(\boldsymbol{x}))}{\partial x_k} = e_{-\boldsymbol{\alpha}}(\boldsymbol{x})\mathcal{F}_\alpha(-iy_k\csc\alpha_kf(\boldsymbol{y}))(\boldsymbol{x}),
$$

for $\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\ldots,\alpha_n)\in\mathbb{R}^n$ with $\alpha_k\notin\pi\mathbb{Z},\ k=1,2,\ldots,n.$

Proof. Let $E_k=(0,\ldots,0,\delta,0,\ldots,0)$, $\delta\neq 0$, *and let δ be the kth variable*. Let $\tilde{\boldsymbol{y}}=(\tilde{y}_1,\ldots,\tilde{y}_n)=(y_1\csc\alpha_1,\ldots,y_n\csc\alpha_n)$. Then

$$
\begin{aligned}
\frac{\partial(e_{-\boldsymbol{\alpha}}(\boldsymbol{x})\mathcal{F}_\alpha(f)(\boldsymbol{x}))}{\partial x_k} &= \lim_{\delta\to 0}\frac{e_{-\boldsymbol{\alpha}}(\boldsymbol{x}+E_k)\mathcal{F}_\alpha(f)(\boldsymbol{x}+E_k)-e_{-\boldsymbol{\alpha}}(\boldsymbol{x})\mathcal{F}_\alpha(f)(\boldsymbol{x})}{\delta}\\
&= \lim_{\delta\to 0}\frac{1}{\delta}\left(\int_{\mathbb{R}^n}\frac{c(\boldsymbol{\alpha})}{\left(\sqrt{2\pi}\right)^n}f(\boldsymbol{y})e_\alpha(\boldsymbol{y})e^{-i(\boldsymbol{x}+E_k)\cdot\tilde{\boldsymbol{y}}}d\boldsymbol{y}\right.\\
&\qquad\left.-\int_{\mathbb{R}^n}\frac{c(\boldsymbol{\alpha})}{\left(\sqrt{2\pi}\right)^n}f(\boldsymbol{y})e_\alpha(\boldsymbol{y})e^{-i\boldsymbol{x}\cdot\tilde{\boldsymbol{y}}}d\boldsymbol{y}\right)\\
&= \lim_{\delta\to 0}\frac{1}{\delta}\left(\int_{\mathbb{R}^n}\frac{c(\boldsymbol{\alpha})}{\left(\sqrt{2\pi}\right)^n}f(\boldsymbol{y})e_\alpha(\boldsymbol{y})e^{-i\boldsymbol{x}\cdot\tilde{\boldsymbol{y}}}(e^{-i\delta\tilde{y}_k}-1)d\boldsymbol{y}\right).
\end{aligned}
$$

By $|\frac{e^{-i\delta\tilde{y}_k}-1}{\delta}|\leq|\tilde{y}_k|$, $x_kf(\boldsymbol{x})\in L^1(\mathbb{R}^n)$ and the Lebesgue dominated convergence theorem we write

$$\begin{aligned}\frac{\partial(e_{-\alpha}(\boldsymbol{x})\mathcal{F}_\alpha(f)(\boldsymbol{x}))}{\partial x_k} &= \int_{\mathbb{R}^n} \frac{c(\boldsymbol{\alpha})}{\left(\sqrt{2\pi}\right)^n} f(\boldsymbol{y}) e_\alpha(\boldsymbol{y}) e^{-i\boldsymbol{x}\cdot\tilde{\boldsymbol{y}}} \left[\lim_{\delta\to 0} \frac{1}{\delta}(e^{-i\delta\tilde{y}_k} - 1)\right] d\boldsymbol{y} \\ &= \int_{\mathbb{R}^n} \frac{c(\boldsymbol{\alpha})}{\left(\sqrt{2\pi}\right)^n} f(\boldsymbol{y}) e_\alpha(\boldsymbol{y}) e^{-i\boldsymbol{x}\cdot\tilde{\boldsymbol{y}}}(-i\tilde{y}_k) d\boldsymbol{y} \\ &= e_{-\alpha}(\boldsymbol{x})\mathcal{F}_\alpha(-iy_k \csc\alpha_k f(\boldsymbol{y}))(\boldsymbol{x}),\end{aligned}$$

This proves the claim. □

3.1. A priori bounds in partial differential equations

We next discuss applications of R_j^α related to the priori bounds.

Theorem 3.3. *Suppose that $f \in S(\mathbb{R}^2)$. Then, we have the a priori bound*

$$\left\|\frac{\partial(e_\alpha f)}{\partial y_1}\right\|_{L^p} + \left\|\frac{\partial(e_\alpha f)}{\partial y_2}\right\|_{L^p} \leq C\left\|e_{-\alpha}\frac{\partial(e_\alpha f)}{\partial y_1} + ie_\alpha\frac{\partial(e_\alpha f)}{\partial y_2}\right\|_{L^p},$$

where $1 < p < \infty$, $\boldsymbol{\alpha} = (\alpha_1, \alpha_2) \in \mathbb{R}^2$ with $\alpha_k \notin \pi\mathbb{Z},\ k = 1, 2$.

To prove Theorem 3.3, we need the following lemma.

Lemma 3.4. *Let $f \in S(\mathbb{R}^2)$ and Let $\boldsymbol{y} = (y_1, y_2) \in \mathbb{R}^2$. We have*

$$e_{-\alpha}(\boldsymbol{y})\frac{\partial(e_\alpha f)}{\partial y_j} = -R_j^\alpha(R_1^\alpha - iR_2^\alpha)\left(e_{-\alpha}\frac{\partial(e_\alpha f)}{\partial y_1} + ie_\alpha\frac{\partial(e_\alpha f)}{\partial y_2}\right)(\boldsymbol{y}),$$

for $j = 1, 2$ and $\boldsymbol{\alpha} = (\alpha_1, \alpha_2) \in \mathbb{R}^2$ with $\alpha_k \notin \pi\mathbb{Z}, k = 1, 2$.

Proof. Taking the FRFT of the above identity, we obtain

$$\begin{aligned}&\mathcal{F}_\alpha\left(-R_j^\alpha(R_1^\alpha - iR_2^\alpha)\left(e_{-\alpha}\frac{\partial(e_\alpha f)}{\partial y_1} + ie_\alpha\frac{\partial(e_\alpha f)}{\partial y_2}\right)\right)(x) \\ &= -\frac{i\tilde{x}_j}{|\tilde{\boldsymbol{x}}|}\left(\frac{i\tilde{\boldsymbol{x}}_1}{|\tilde{\boldsymbol{x}}|} + \frac{i\tilde{\boldsymbol{x}}_2}{|\tilde{\boldsymbol{x}}|}\right)(ix_1\csc\alpha_1\mathcal{F}_\alpha(f)(\boldsymbol{x}) - x_2\csc\alpha_2\mathcal{F}_\alpha(f)(\boldsymbol{x})) \\ &= -\frac{i\tilde{x}_j}{|\tilde{\boldsymbol{x}}|}\left(\frac{-|\tilde{x}_1|^2}{|\tilde{\boldsymbol{x}}|} + \frac{-|\tilde{x}_2|^2}{|\tilde{\boldsymbol{x}}|}\right)\mathcal{F}_\alpha(f)(\boldsymbol{x}) \\ &= ix_j\csc\alpha_j\mathcal{F}_\alpha(f)(\boldsymbol{x}).\end{aligned}$$

By Lemma 3.1, we have

$$\mathcal{F}_\alpha\left(e_{-\alpha}\frac{\partial(e_\alpha f)}{\partial y_j}\right)(\boldsymbol{x}) = ix_j\csc\alpha_j\mathcal{F}_\alpha(f)(\boldsymbol{x}).$$

Applying the inverse FRFT on the above identity we deduce the desired result. □

Now, we return to prove Theorem 3.3.

Proof. By Lemma 3.1 and Theorem 2.4, we have

$$\begin{aligned}\left\|\frac{\partial(e_\alpha f)}{\partial y_j}\right\|_{L^p} &= \left\| -R_j^\alpha(R_1^\alpha - iR_2^\alpha)\left(e_{-\alpha}\frac{\partial(e_\alpha f)}{\partial y_1} + ie_\alpha\frac{\partial(e_\alpha f)}{\partial y_2}\right)\right\|_{L^p} \\ &\le C\left\|e_{-\alpha}\frac{\partial(e_\alpha f)}{\partial y_1} + ie_\alpha\frac{\partial(e_\alpha f)}{\partial y_2}\right\|_{L^p},\end{aligned}$$

which completes the proof of the theorem. □

Remark 3.1. When $\alpha = (\frac{\pi}{2} + k_1\pi, \frac{\pi}{2} + k_2\pi, \ldots, \frac{\pi}{2} + k_n\pi), k_j \in \mathbb{Z}$ for $j = 1, 2, \ldots, m$, Theorem 3.3 reduces to Proposition 4 in [23, pp.60].

In the sequel, $\Delta = \sum_{j=1}^n \frac{\partial^2}{\partial y_j^2}$ denotes the usual Laplacian on $\mathbb{R}^n$.

Lemma 3.5. *For $f \in S(\mathbb{R}^n)$, $1 \le j, k \le n$, and $\boldsymbol{x} \in \mathbb{R}^n$ we have*

$$e_{-\alpha}(\boldsymbol{x})\frac{\partial^2(e_\alpha(\boldsymbol{x})f(\boldsymbol{x}))}{\partial x_k x_j} = \left(-R_k^\alpha R_j^\alpha e_{-\alpha}\Delta(e_\alpha f)\right)(\boldsymbol{x}),$$

where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n$ with $\alpha_k \notin \pi\mathbb{Z}$.

Proof. Applying the FRFT on the expression on the left we write

$$\begin{aligned}\mathcal{F}_\alpha\left(e_{-\alpha}\frac{\partial^2(e_\alpha f)}{\partial y_k\partial y_j}\right)(\boldsymbol{x}) &= \mathcal{F}_\alpha\left(e_{-\alpha}\frac{\partial\left(e_\alpha\frac{e_{-\alpha}\partial(e_\alpha f)}{\partial y_j}\right)}{\partial y_k}\right)(\boldsymbol{x}) \\ &= ix_k\csc\alpha_k\mathcal{F}_\alpha\left(e_{-\alpha}\frac{\partial[e_\alpha f]}{\partial y_j}\right)(\boldsymbol{x}) \\ &= ix_k\csc\alpha_k ix_j\csc\alpha_j\mathcal{F}_\alpha(f)(\boldsymbol{x}) \\ &= -\left(\frac{i\tilde{x}_k}{|\tilde{\boldsymbol{x}}|}\right)\left(\frac{i\tilde{x}_j}{|\tilde{\boldsymbol{x}}|}\right)(-|\tilde{\boldsymbol{x}}|^2)\mathcal{F}_\alpha(f)(\boldsymbol{x}) \\ &= -\left(\frac{i\tilde{x}_k}{|\tilde{\boldsymbol{x}}|}\right)\left(\frac{i\tilde{x}_j}{|\tilde{\boldsymbol{x}}|}\right)\mathcal{F}_\alpha\left(e_{-\alpha}\frac{\partial^2(e_\alpha f)}{\partial^2 y_1} + \cdots + e_{-\alpha}\frac{\partial^2(e_\alpha f)}{\partial^2 y_n}\right)(\boldsymbol{x}) \\ &= \mathcal{F}_\alpha(-R_k^\alpha R_j^\alpha e_{-\alpha}\Delta(e_\alpha f))(\boldsymbol{x}).\end{aligned}$$

Applying the inverse FRFT on the above identity, we obtain the desired result. □

Remark 3.2. When $\alpha = (\frac{\pi}{2} + k_1\pi, \frac{\pi}{2} + k_2\pi, \ldots, \frac{\pi}{2} + k_n\pi), k_j \in \mathbb{Z}$ for $j = 1, 2, \ldots, m$, Lemma 3.5 reduces to Proposition 5.1.17 in [10].

Theorem 3.6. *Suppose $f \in S(\mathbb{R}^n)$ and $\Delta(e_\alpha f) = \sum_{j=1}^{n} \frac{\partial^2(e_\alpha f)}{\partial y_j^2}$. Then we have the priori bound*

$$\left\|\frac{\partial^2(e_\alpha f)}{\partial y_k \partial y_j}\right\|_{L^p} \leq C \left\|\Delta(e_\alpha f)\right\|_{L^p},$$

for $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n$ with $\alpha_k \notin \pi\mathbb{Z}$, $k = 1, 2, \ldots, n$.

Proof. According to Lemma 3.5 and Theorem 2.4, we obtain that

$$\begin{aligned}
\left\|\frac{\partial^2(e_\alpha f)}{\partial y_k \partial y_j}\right\|_{L^p} &= \left\|e_{-\alpha}\frac{\partial^2(e_\alpha f)}{\partial y_k \partial y_j}\right\|_{L^p} \\
&= \left\|-R_j^\alpha R_k^\alpha e_{-\alpha}\Delta(e_\alpha f)\right\|_{L^p} \\
&\leq C\|\Delta(e_\alpha f)\|_{L^p},
\end{aligned}$$

which completes the proof of the theorem. □

3.2. Laplace's equation with a chirp

Next we we study a version of Laplace's equation.

Example 3.1. Let $f \in L^2(\mathbb{R}^n)$ and suppose that $u \in S'(\mathbb{R}^n)$ satisfies the following form of Laplace's equation with a chirp

$$\Delta(e_\alpha u) = e_\alpha f. \tag{3.1}$$

We express all second-order derivatives of u in terms of the fractional Riesz transform of f. A similar approach can be found in [6]. In order to accomplish this, we make use of the following lemma.

Lemma 3.7. ([10]) *Suppose that $u \in S'(\mathbb{R}^n)$. If $\hat{u}$ is supported at $\{0\}$, then u is a polynomial.*

To solve equation (3.1), we first show that if u satisfies (3.1), then the tempered distribution

$$\mathcal{F}_\alpha(e_{-\alpha}\partial_j\partial_k(e_\alpha u) + R_j^\alpha R_k^\alpha f)$$

is supported at $\{0\}$. Then from Lemma 3.7, we obtain that

$$e_{-\alpha}\partial_j\partial_k(e_\alpha u) = -R_j^\alpha R_k^\alpha f + e_{-\alpha}P,$$

where P is a polynomial of n variables (that depends on j and k). This provides a way of expressing the mixed partial derivatives of $e_\alpha u$ in terms of the fractional Riesz transforms of f.

To prove that the tempered distribution $\mathcal{F}_\alpha(e_{-\alpha}\partial_j\partial_k(e_\alpha u) + R_j^\alpha R_k^\alpha f)$ is supported at $\{0\}$, we pick $\gamma \in S(\mathbb{R}^n)$ whose support does not contain the origin. Then, γ vanishes in a neighborhood of zero. Fix $\eta \in C^\infty$, which is equal to 1 on the support of γ and vanishes in a smaller neighborhood of zero. Define

$$\zeta(\boldsymbol{\xi}) = -\eta(\boldsymbol{\xi})\left(-\frac{i\tilde{\xi}_j}{|\tilde{\boldsymbol{\xi}}|}\right)\left(-\frac{i\tilde{\xi}_k}{|\tilde{\boldsymbol{\xi}}|}\right), \qquad \boldsymbol{\xi} \in \mathbb{R}^n,$$

and we notice that ζ and all of its derivatives are both bounded C^∞ functions. Additionally

$$\eta(\boldsymbol{\xi})(i\tilde{\xi}_j)(i\tilde{\xi}_k) = \zeta(\boldsymbol{\xi})(-|\tilde{\boldsymbol{\xi}}|^2), \qquad \boldsymbol{\xi} \in \mathbb{R}^n.$$

Taking the FRFT of both side of (3.1) we obtain that

$$\mathcal{F}_\alpha(e_{-\alpha}\Delta(e_\alpha u))(\boldsymbol{\xi}) = -|\tilde{\boldsymbol{\xi}}|^2\mathcal{F}_\alpha(u)(\boldsymbol{\xi}) = \mathcal{F}_\alpha(f)(\boldsymbol{\xi}).$$

Multiplying by ζ, we write

$$\zeta(\boldsymbol{\xi})\mathcal{F}_\alpha(e_{-\alpha}\Delta(e_\alpha u))(\boldsymbol{\xi}) = -\zeta(\boldsymbol{\xi})|\tilde{\boldsymbol{\xi}}|^2\mathcal{F}_\alpha(u)(\boldsymbol{\xi}) = \zeta(\boldsymbol{\xi})\mathcal{F}_\alpha(f)(\boldsymbol{\xi}).$$

Since for all $1 \le j, k \le n$,

$$\begin{aligned}
\langle \mathcal{F}_\alpha(e_{-\alpha}\partial_j\partial_k(e_\alpha u)), \gamma\rangle &= \langle (i\tilde{\xi}_j)(i\tilde{\xi}_k)\mathcal{F}_\alpha(u), \gamma\rangle \\
&= \langle (i\tilde{\xi}_j)(i\tilde{\xi}_k)\mathcal{F}_\alpha(u), \eta\gamma\rangle \\
&= \langle \eta(\boldsymbol{\xi})(i\tilde{\xi}_j)(i\tilde{\xi}_k)\mathcal{F}_\alpha(u), \gamma\rangle \\
&= \langle \zeta(\boldsymbol{\xi})(-|\tilde{\boldsymbol{\xi}}|^2)\mathcal{F}_\alpha(u), \gamma\rangle \\
&= \langle \zeta(\boldsymbol{\xi})\mathcal{F}_\alpha(f), \gamma\rangle \\
&= \left\langle -\eta(\boldsymbol{\xi})\left(-i\tilde{\xi}_j/|\tilde{\boldsymbol{\xi}}|\right)\left(-i\tilde{\xi}_k/|\tilde{\boldsymbol{\xi}}|\right)\mathcal{F}_\alpha, \gamma\right\rangle \\
&= \langle -\eta(\boldsymbol{\xi})\mathcal{F}_\alpha(R_j^\alpha R_k^\alpha(f)), \gamma\rangle \\
&= -\langle \mathcal{F}_\alpha(R_j^\alpha R_k^\alpha(f)), \eta\gamma\rangle \\
&= -\langle \mathcal{F}_\alpha(R_j^\alpha R_k^\alpha(f)), \gamma\rangle,
\end{aligned}$$

and since the support of $\gamma \in S(\mathbb{R}^n)$ does not contain the origin, it follows that the function $\mathcal{F}_\alpha(e_{-\alpha}\partial_j\partial_k(e_\alpha u) + R_j^\alpha R_k^\alpha(f))$ is supported at $\{0\}$.

4. Numerical simulation of fractional multipliers

In this section, we apply the fractional Riesz transform to an image with the help of the FRFT discrete algorithm ([1, 2, 27]).

As shown in Fig. 4.1, (a) is the original 2-dimensional grayscale image with 400 pixels × 400 pixels; (c) is the 2-dimensional grayscale image after the Riesz transform of order $(\pi/4, \pi/4)$. In the continuous case, Fig. 4.1 (a) can be regarded as a function of $\mathbb{R}^2$

$$f(x_1, x_2) = \begin{cases} 0, & (x_1, x_2) \in [0, 200]^2 \cup [200, 400]^2, \\ 255, & \text{otherwise.} \end{cases}$$

Fig. 4.1 (b) and (d) are the 3-dimensional color graphs of f and $R_{x_1}^{(\pi/4,\pi/4)} f$.

Recall that the fractional Fourier multiplier of the fractional Riesz transform R_j^α is

$$m_j^\alpha(\boldsymbol{u}) := -i\frac{\tilde{u}_j}{|\tilde{\boldsymbol{u}}|}.$$

Graphs (a) and (b) in Fig. 4.2 indicate that the fractional Riesz transform has the effect of amplitude reduction.

By comparing (c)/(e) and (d)/(f) in Fig. 4.2, as well as the real/imaginary part of $\mathcal{F}_\alpha f$ and the imaginary/real part of $\mathcal{F}_\alpha(R_j^\alpha f)$ accordingly, it can be seen that the fractional Riesz transform has the effect of phase shifting.

Above all, Fig. 4.2 shows that the fractional multiplier of R_j^α is $-i\tilde{u}_j/|\tilde{\boldsymbol{u}}|$, and thus, the fractional Riesz transform is not only a phase-shift converter but also an amplitude attenuator. This circumstance is quite different from that of the fractional multiplier of H_α, which is only a phase-shift converter with multiplier $-i\text{sgn}((\pi - \alpha)u)$.

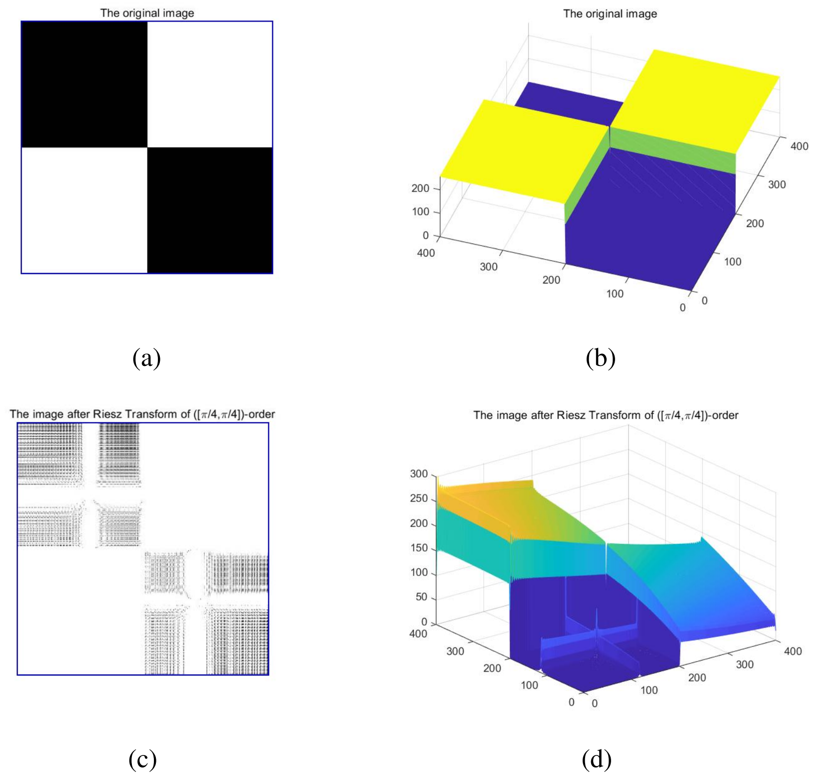


Figure 4.1: Fractional Riesz transform of order $(\pi/4, \pi/4)$ of a image.

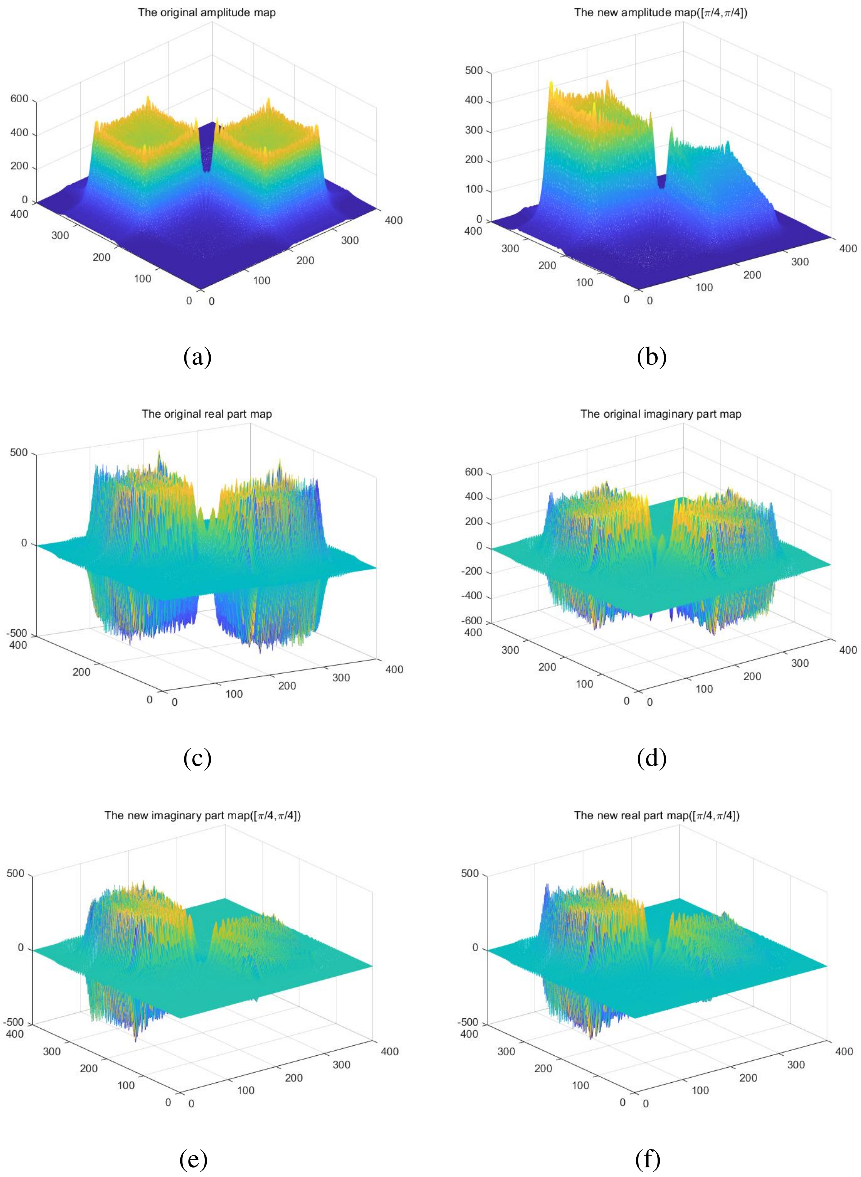


Figure 4.2: Phase-shifting and amplitude-reducing effect of the fractional Riesz transform in the fractional Fourier domain of order $\alpha = (\pi/4, \pi/4)$ on an image.

5. Application of the fractional Riesz transforms in edge detection

Edge detection is a key technology in image processing. It is widely used in biometrics, image understanding, visual attention and other fields. Commonly used image feature extraction methods include the Roberts operator, Prewitt operator, Sobel operator, Laplacian operator and Canny operator. These algorithms extract features based on the gradient changes in the pixel amplitudes. In [16, 33, 34], the authors introduced the image edge detection methods based on the Riesz transform, which can avoid the influence of uneven illumination. Moreover, the Riesz transform has the characteristics of isotropy; therefore, the Riesz transform has more advantages in feature extraction. Based on the principle of Riesz transform edge detection, edge detection based on the *fractional Riesz transform* is proposed in this section.

When processing the two-dimensional signal $f(\boldsymbol{x})$, the fractional Riesz transform of $f(\boldsymbol{x})$ can be expressed as

$$(R_1^\alpha f)(\boldsymbol{x}) = c_n \text{ p.v. } e_{-\alpha}(\boldsymbol{x})\left((e_\alpha f) * \frac{x_1}{|\boldsymbol{x}|^3}\right)(\boldsymbol{x}), \tag{5.1}$$

$$(R_2^\alpha f)(\boldsymbol{x}) = c_n \text{ p.v. } e_{-\alpha}(\boldsymbol{x})\left((e_\alpha f) * \frac{x_2}{|\boldsymbol{x}|^3}\right)(\boldsymbol{x}), \tag{5.2}$$

or

$$(R_1^\alpha f)(\boldsymbol{x}) = \left[\mathcal{F}_{-\alpha}\left(-i\frac{u_1 \csc \alpha_1}{\sqrt{(u_1 \csc \alpha_1)^2 + (u_2 \csc \alpha_2)^2}}(\mathcal{F}_\alpha f)(\boldsymbol{u})\right)\right](\boldsymbol{x}), \tag{5.3}$$

$$(R_2^\alpha f)(\boldsymbol{x}) = \left[\mathcal{F}_{-\alpha}\left(-i\frac{u_2 \csc \alpha_2}{\sqrt{(u_1 \csc \alpha_1)^2 + (u_2 \csc \alpha_2)^2}}(\mathcal{F}_\alpha f)(\boldsymbol{u})\right)\right](\boldsymbol{x}). \tag{5.4}$$

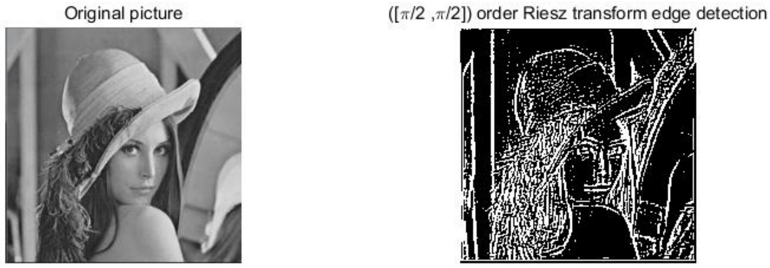
For an image $f(\boldsymbol{x})$, the monogenic signal is defined as the combination of $f(\boldsymbol{x})$ and its fractional Riesz transform.

$$(p(\boldsymbol{x}), q_1(\boldsymbol{x}), q_2(\boldsymbol{x})) = (f(\boldsymbol{x}), (R_1^\alpha f)(\boldsymbol{x}), (R_2^\alpha f)(\boldsymbol{x})).$$

Therefore, the local amplitude value $A(\boldsymbol{x})$, local orientation $\theta(\boldsymbol{x})$ and local phase $P(\boldsymbol{x})$ in the monogenic signal in the image can be expressed as

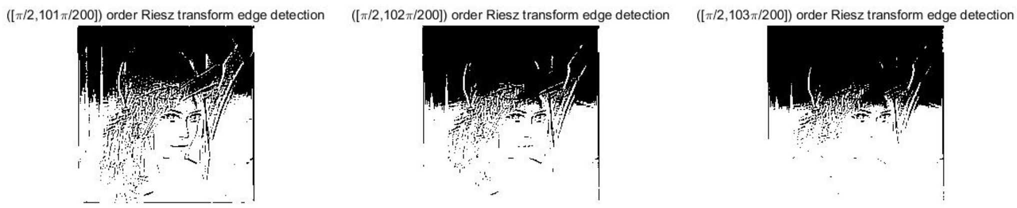
$$\begin{cases} A(x) = \sqrt{p(\boldsymbol{x})^2 + |q_1(\boldsymbol{x})|^2 + |q_2(\boldsymbol{x})|^2} \\ \theta(x) = \tan^{-1}\left(\left|\dfrac{q_2(\boldsymbol{x})}{q_1(\boldsymbol{x})}\right|\right) \\ P(x) = \tan^{-1}\left(\dfrac{p(\boldsymbol{x})}{\sqrt{|q_1(\boldsymbol{x})|^2 + |q_2(\boldsymbol{x})|^2}}\right). \end{cases}$$

In this paper, we use the fractional Riesz transform in the form of (5.3) and (5.4) for algorithm design because the fractional Riesz transform can be decomposed into a combination of a FRFT, inverse FRFT and multiplier. Because the FRFT and inverse FRFT have fast algorithms, compared with the form of (5.1) and (5.2), the computational complexity of the algorithm in the form of (5.3) and (5.4) is reduced. Thus, a faster and more efficient operation is realized.

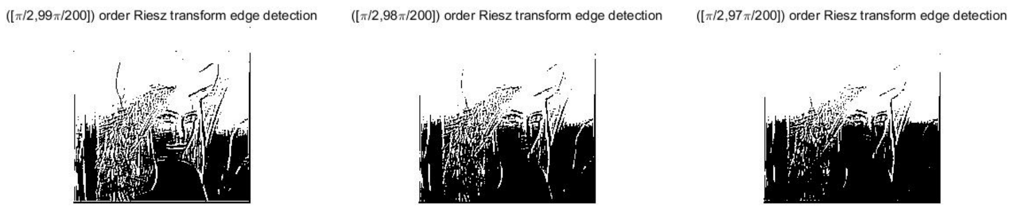


(a) (b)

Figure 5.1: Original image and edge detection based on the fractional Riesz transform of order ($[\pi/2, \pi/2]$) (i.e., the classical Riesz transform)



(a) (b) (c)



(d) (e) (f)

Figure 5.2: Extract the information of a specific position in the lateral direction of the image

The simulation experiment will be conducted by using the classical Lena image ((a) in Fig. 5.1) as the test image. To better highlight the results of our simulation experiment, we choose an appropriate threshold value to binarize the images after the experiment in such a way that our experimental results show a more obvious effect. Graph (b) in Fig. 5.1 illustrates the result of edge detection based on the fractional Riesz transform of order ($[\pi/2, \pi/2]$) (i.e., the classical Riesz transform).

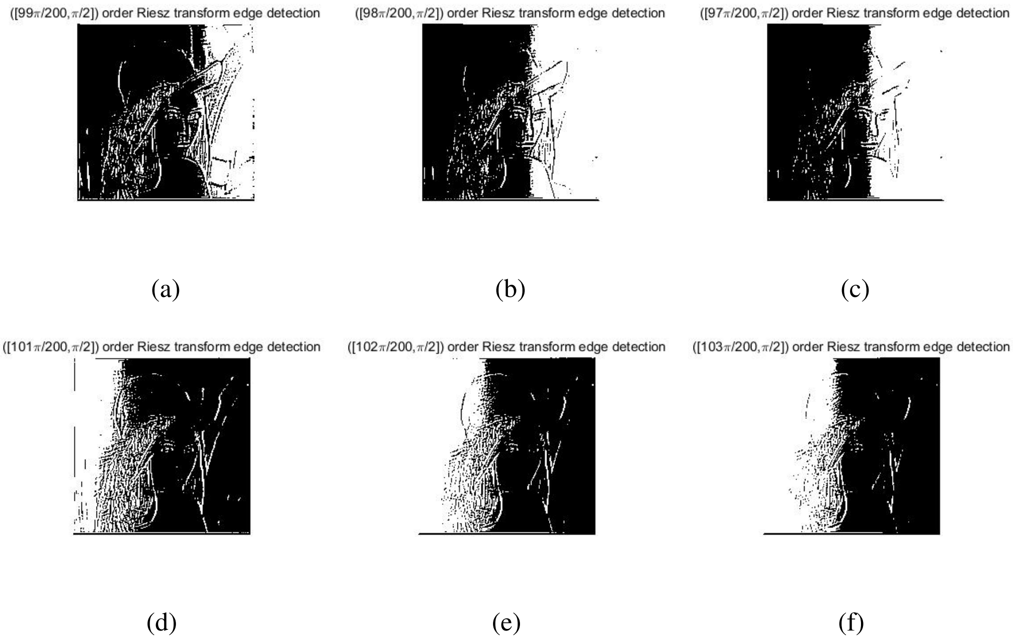


Figure 5.3: Extract the information of specific position in the longitudinal direction of the image

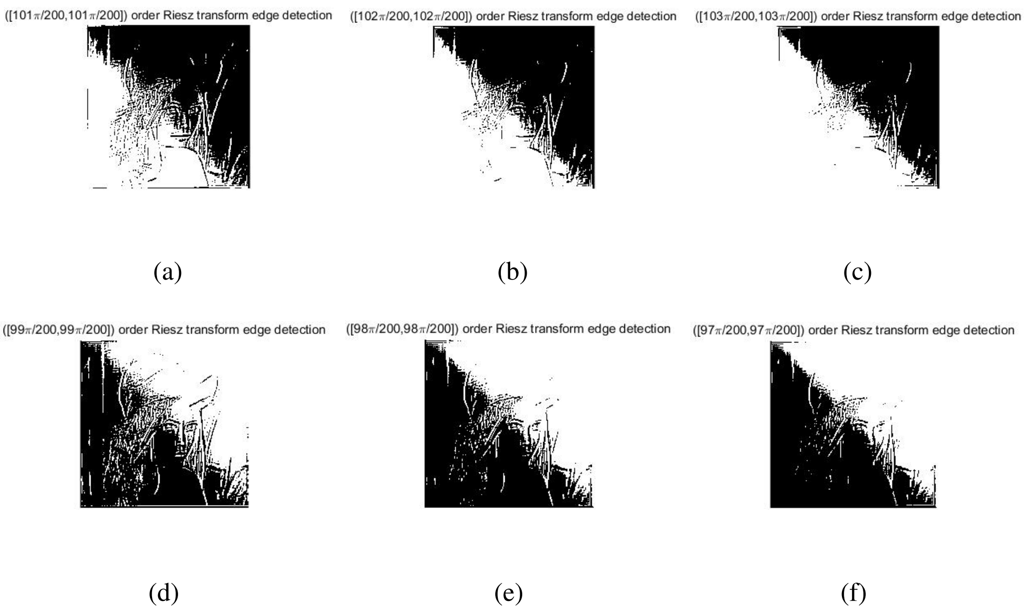


Figure 5.4: Extract the information of the specific position in the main diagonal of the image

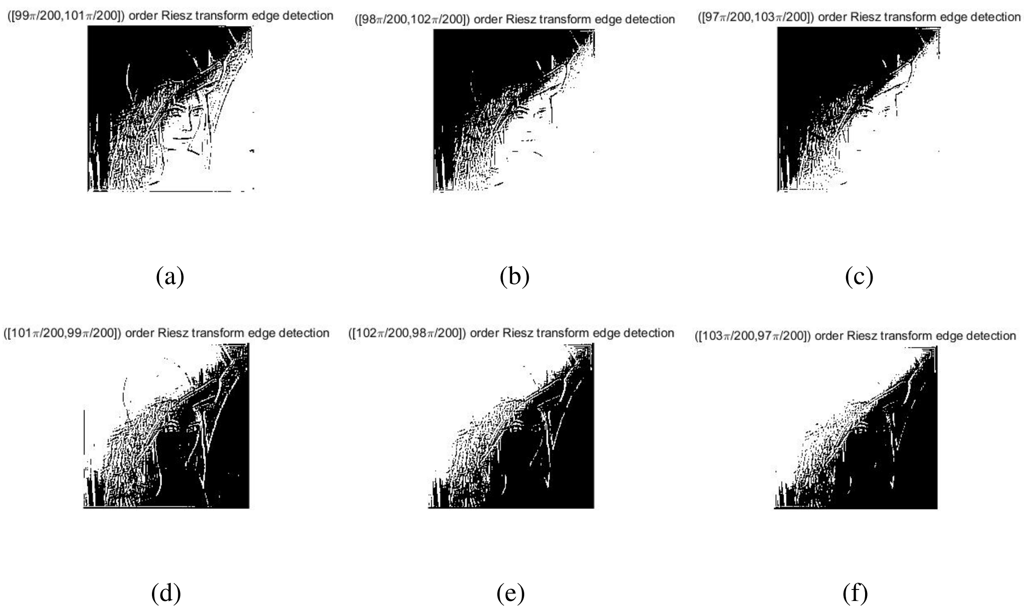


Figure 5.5: Extract the information of the specific position in the counter diagonal of the image

Fig. 5.2 shows that when $\alpha_1 = \pi/2$ is fixed, if α_2 decreases from $\pi/2$ ((d), (e), (f) in Fig. 5.2), we extract information about the lateral up position. If α_2 increases from $\pi/2$ ((a), (b), (c) in Fig. 5.2), we extract information about the lateral down position. In conclusion, by fixed $\alpha_1 = \pi/2$, we can adjust α_2 to extract information on the specific position in the lateral direction.

Fig. 5.3 indicates that when $\alpha_2 = \pi/2$ is fixed and α_1 decreases from $\pi/2$ ((a), (b), (c) in Fig. 5.3), the longitudinal right position information is extracted. As α_1 increases from $\pi/2$ ((d), (e), (f) in Fig. 5.3), we extract the longitudinal left position information. In conclusion, when fixing $\alpha_2 = \pi/2$, we can extract information of the specific longitudinal positions by adjusting α_1.

Fig. 5.4 shows that when α_1 and α_2 increase or decrease the same size from $\pi/2$, the information on the specific direction in the main diagonal is extracted.

Fig. 5.5 indicates that when α_1 increases from $\pi/2$ and α_2 decreases from $\pi/2$ by the same size, the information on the specific direction in the antidiagonal is extracted.

The preceding simulation provides a new edge detection tool based on the the fractional Riesz transform, that extracts both global features and local features of images. This numerical implementation confirms the belief expressed in [26] that amplitude, phase, and direction information can be simultaneously extracted by controlling the order of the fractional Riesz transform. We predict that very comprehensive analysis and processing of multidimensional signals, such as images, videos, 3D meshes and animations, can be achieved via the fractional Riesz transforms.

6. Chirp Hardy spaces

In this section, we naturally consider the boundedness of a singular integral operator with a chirp function on non-rotation invariant space, such as Hardy spaces. Hardy spaces are spaces of distributions which become more singular as p decreases and can be regarded as a substitute for L^p when $p < 1$.

6.1. Chirp Hardy spaces and chirp BMO spaces

We recall the real variable characterization and atom characterization of Hardy spaces.

Definition 6.1. ([4, 10]) *Let f be a bounded tempered distribution on $\mathbb{R}^n$ and let $0 < p < \infty$. We say that f lies in the Hardy spaces $H^p(\mathbb{R}^n)$ if the Poisson maximal function*

$$M(f;P)(\boldsymbol{x}) = \sup_{t>0} |(P_t * f)(\boldsymbol{x})|$$

lies in $L^p(\mathbb{R}^n)$, where the Poisson kernel P is the function

$$P(\boldsymbol{x}) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \frac{1}{(1+|\boldsymbol{x}|^2)^{\frac{n+1}{2}}}.$$

For $t > 0$, let $P_t(\boldsymbol{x}) = t^{-n}P(t^{-1}\boldsymbol{x})$. If this is the case, we set

$$\|f\|_{H^p} = \|M(f;P)\|_{L^p}.$$

Remark 6.1. Suppose that $f \in S(\mathbb{R}^n)$. We have

$$\begin{aligned}
\|f\|_{H^p} &= \left\| \sup_{t>0} |(P_t * f)| \right\|_{L^p} = \left\| \sup_{t>0} \left| \int_{\mathbb{R}^n} P_t(\cdot - \boldsymbol{y}) f(\boldsymbol{y}) d\boldsymbol{y} \right| \right\|_{L^p}, \\
\|e_\alpha f\|_{H^p} &= \left\| \sup_{t>0} |(P_t * (e_\alpha f))| \right\|_{L^p} = \left\| \sup_{t>0} \left| \int_{\mathbb{R}^n} P_t(\cdot - \boldsymbol{y}) e_\alpha(\boldsymbol{y}) f(\boldsymbol{y}) d\boldsymbol{y} \right| \right\|_{L^p}.
\end{aligned}$$

We can clearly see that $\|e_\alpha f\|_{H^p}$ depends on $\boldsymbol{\alpha}$, that is,

$$\|f\|_{H^p} \neq \|e_\alpha f\|_{H^p}.$$

Note that $H^p(\mathbb{R}^n)$ is not a rotation-invariant space.

Now let us consider the boundedness of singular integral operators with chirp functions in Hardy space when kernel K satisfies certain size and smoothness conditions. Let us recall the definition of the δ-Calderón-Zygmund operator.

Definition 6.2. ([17]) *Let T be a bounded linear operator. We say that T is a δ-Calderón-Zygmund operator if T is bounded on $L^2(\mathbb{R}^n)$ and K is a continuous function* *on* $\mathbb{R}^n \times \mathbb{R}^n \setminus \{(\boldsymbol{x},\boldsymbol{y}) : \boldsymbol{x} \neq \boldsymbol{y}\}$ *that satisfies*

(1) $|K(\boldsymbol{x},\boldsymbol{y})| \leq \frac{C}{|\boldsymbol{x}-\boldsymbol{y}|^n}, \ \boldsymbol{x} \neq \boldsymbol{y}$;

(2) $|K(\boldsymbol{x},\boldsymbol{y}) - K(\boldsymbol{x},\boldsymbol{z})| + |K(\boldsymbol{y},\boldsymbol{x}) - K(\boldsymbol{z},\boldsymbol{x})| \leq C\frac{|\boldsymbol{y}-\boldsymbol{z}|^\delta}{|\boldsymbol{x}-\boldsymbol{z}|^{n+\delta}}, \quad \text{if } |\boldsymbol{x}-\boldsymbol{z}| > 2|\boldsymbol{y}-\boldsymbol{z}|, \ 0 < \delta \leq 1$;

(3) For $f, g \in S(R^n)$ *and* $\operatorname{supp} f \cap \operatorname{supp} g = \emptyset$*, one has*

$$(Tf, g) = \int K(\boldsymbol{x},\boldsymbol{y}) f(\boldsymbol{y}) g(\boldsymbol{x}) d\boldsymbol{y} d\boldsymbol{x}.$$

Lemma 6.3. ([17]) *Suppose that T is a δ-Calderón-Zygmund operator and its conjugate operator $T^* = 0$. Then, T can be extended to the bounded operator from $H^p(\mathbb{R}^n)$ to $H^p(\mathbb{R}^n)$, where $0 < \delta \leq 1$ and $\frac{n}{n+\delta} < p \leq 1$.*

By standard calculations, we have the following estimates for K^α:

$$|K^\alpha(\boldsymbol{x},\boldsymbol{y}) - K^\alpha(\boldsymbol{x},\boldsymbol{z})| \leq |K(\boldsymbol{x},\boldsymbol{y}) - K(\boldsymbol{x},\boldsymbol{z})| + L_\alpha(\boldsymbol{y},\boldsymbol{z})|K(\boldsymbol{x},\boldsymbol{z})||\boldsymbol{y}-\boldsymbol{z}|,$$

where

$$L_\alpha(\boldsymbol{y},\boldsymbol{z}) = |\nabla e_\alpha(\boldsymbol{w})| = \sqrt{\sum_{k=1}^{n} |e_\alpha(\boldsymbol{w}) \cot \alpha_k w_k|^2},$$

and $\boldsymbol{w} = \boldsymbol{z} + (\theta_1(y_1 - z_1), \ldots, \theta_n(y_n - z_n)))$ for $\theta_j \in (0,1)$. It is known that for $\boldsymbol{\alpha} = (\frac{\pi}{2} + k_1\pi, \frac{\pi}{2} + k_2\pi, \ldots, \frac{\pi}{2} + k_n\pi)$, K^α satisfies the δ-Calderón-Zygmund operator kernel condition (2). It is obvious that K^α satisfies (1) in Definition 6.2, but (2) in Definition 6.2 is not guaranteed.

We now define a new class of Hardy space with chirp functions.

Definition 6.4. *Let f be a bounded tempered distribution on $\mathbb{R}^n$ and let $0 < p < \infty$. We say that f lies in the chirp Hardy space $H^p_\alpha(\mathbb{R}^n)$ for $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ with $\alpha_k \notin \pi\mathbb{Z}$ for all $k = 1, 2, \ldots, n$, if the Poisson maximal function with chirp function*

$$M_\alpha(f; P) = \sup_{t>0} |(P_t * (e_\alpha f))|$$

lies in $L^p(\mathbb{R}^n)$. If this is the case, we set

$$\|f\|_{H^p_\alpha} = \|M_\alpha(f; P)\|_{L^p}.$$

Lemma 6.5. ([24]) *Suppose $f \in L^P(\mathbb{R}^n)$, $p > 1$, and f has bounded support. Then $f \in H^1(\mathbb{R}^n)$ if and only if $\int_{\mathbb{R}^n} f(x)dx = 0$.*

Remark 6.2. We provide an example indicating that $H^1_\alpha(\mathbb{R})$ is not contained in $H^1(\mathbb{R})$. Let $f(x) = e_{-\alpha}(x)e^{-ix}\chi_{[-\pi,\pi]}(x)$ with $\alpha_k \notin \frac{\pi}{2}\mathbb{Z}$ for all $k = 1, 2, \ldots, n$. From $f \in H^1_\alpha(\mathbb{R})$ if and only if $e_\alpha f \in H^1(\mathbb{R})$ and by Lemma 6.5 we infer that

$$\int_{\mathbb{R}} e_\alpha e_{-\alpha}(x) e^{-ix} \chi_{[-\pi,\pi]}(x)\, dx = 0,$$

and

$$\int_{\mathbb{R}} e_{-\alpha}(x) e^{-ix} \chi_{[-\pi,\pi]}(x)\, dx \neq 0,$$

namely, $f \in H^1_\alpha(\mathbb{R})$ and $f \notin H^1(\mathbb{R})$.

Lemma 6.6. ([11]) *The Hardy space $H^p(\mathbb{R}^n)$ is a complete space.*

Theorem 6.7. *The chirp Hardy space $H^p_\alpha(\mathbb{R}^n)$ is a complete space.*

Proof. Let $\{f_k\}$ be a Cauchy sequence in $H^p_\alpha(\mathbb{R}^n)$. Then, $\{e_\alpha f_k\}$ is a Cauchy sequence in $H^p(\mathbb{R}^n)$. By Lemma 6.6, there exists an $\bar{f} \in H^p(\mathbb{R}^n)$ such that

$$\lim_{k\to\infty} \|e_\alpha f_k - \bar{f}\|_{H^p} = 0.$$

The above identity is rewritten as

$$\lim_{k\to\infty} \|f_k - e_{-\alpha}\bar{f}\|_{H^p_\alpha} = 0.$$

Since $\bar{f} \in H^p(\mathbb{R}^n)$, we obtain $f := e_{-\alpha}\bar{f} \in H^p_\alpha(\mathbb{R}^n)$, which completes the proof of the theorem. □

It is known that the dual space of H^1 is the BMO space. To study the dual space of the chirp Hardy space, we define a new BMO space with a chirp function as follows.

Definition 6.8. *Suppose that f is a locally integrable function on $\mathbb{R}^n$. Define the chirp BMO space or BMO^α as*

$$BMO^\alpha(\mathbb{R}^n) = \{f : \|f\|_{BMO^\alpha} < \infty\}.$$

Let

$$\|f\|_{BMO^\alpha} = \sup_Q \frac{1}{|Q|} \int_Q |e_\alpha(\boldsymbol{x}) f(\boldsymbol{x}) - \mathrm{Avg}_Q(e_\alpha f)| d\boldsymbol{x},$$

where the supremum is taken over all cubes Q in $\mathbb{R}^n$ and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ with $\alpha_k \notin \pi\mathbb{Z}$, $k = 1, 2, \ldots, n$.

Lemma 6.9. ([7, 11]) *BMO is a complete space.*

Theorem 6.10. *BMO^α is complete.*

The proof follows the same pattern as that of Theorem 6.7 and is based on Lemma 6.9.

6.2. Dual spaces of chirp Hardy spaces

Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n$ with $\alpha_k \notin \pi\mathbb{Z}$ for all $k = 1, 2, \ldots, n$. We discuss the dual spaces of chirp Hardy spaces $H^p_{\boldsymbol{\alpha}}(\mathbb{R}^n)$ for $0 < p \leq 1$. When $p = 1$, we have the following theorem.

Theorem 6.11. $(H^1_{\boldsymbol{\alpha}})^*(\mathbb{R}^n) = BMO^{-\boldsymbol{\alpha}}(\mathbb{R}^n)$.

Lemma 6.12. ([7], [8], [9]) $(H^1)^*(\mathbb{R}^n) = BMO(\mathbb{R}^n)$.

Now, we will go back to prove Theorem 6.11.

Proof. For any $f \in H^1_{\boldsymbol{\alpha}}(\mathbb{R}^n)$ and $g \in (H^1_{\boldsymbol{\alpha}})^*(\mathbb{R}^n)$, one has

$$\langle f, g\rangle = \langle e_{-\boldsymbol{\alpha}} e_{\boldsymbol{\alpha}} f, g\rangle = \langle e_{\boldsymbol{\alpha}} f, e_{-\boldsymbol{\alpha}} g\rangle.$$

From $f \in H^1_{\boldsymbol{\alpha}}(\mathbb{R}^n)$ if and only if $e_{\boldsymbol{\alpha}} f \in H^1(\mathbb{R}^n)$ and Lemma 6.12, we deduce that, $e_{-\boldsymbol{\alpha}} g$ lies in $BMO(\mathbb{R}^n)$, therefore $g \in BMO^{-\boldsymbol{\alpha}}(\mathbb{R}^n)$, completing the proof of the theorem. $\square$

Remark 6.3. When $\boldsymbol{\alpha} = (\frac{\pi}{2} + k_1\pi, \frac{\pi}{2} + k_2\pi, \ldots, \frac{\pi}{2} + k_n\pi), k_j \in \mathbb{Z}$ for $j = 1, 2, \ldots, m$, $(H^1_{\boldsymbol{\alpha}})^*(\mathbb{R}^n) = BMO^{-\boldsymbol{\alpha}}(\mathbb{R}^n)$ reduces to $(H^1)^*(\mathbb{R}^n) = BMO(\mathbb{R}^n)$.

We proceed by considering the dual space of the chirp Hardy space when $0 < p < 1$.

Lemma 6.13. ([25]) *For $g \in L^1_{loc}(\mathbb{R}^n)$, Q is an any cube in $\mathbb{R}^n$ and $s \in \mathbb{Z}^+$. Then there exists a unique polynomial $P_Q(g)$ whose degree does not exceed s that satisfies*

$$\int_Q [g(\boldsymbol{x}) - P_Q(g)(\boldsymbol{x})]\boldsymbol{x}^\alpha d\boldsymbol{x} = 0, \quad 0 \leq |\alpha| \leq s.$$

Definition 6.14. ([25]) *For $s \in \mathbb{Z}^+$, $0 \leq [n\beta] \leq s$ and $1 \leq q' \leq \infty$, the Campanato-Meyers space $L(\beta, q', s)(\mathbb{R}^n)$ is defined as the set of locally integrable functions g that satisfy*

$$||g||_{L(\beta,q',s)} = \sup_{Q \subset \mathbb{R}^n} |Q|^{-\beta} \left[\int_Q |g(\boldsymbol{x}) - P_Q(g)(\boldsymbol{x})|^{q'} \frac{d\boldsymbol{x}}{|Q|}\right]^{\frac{1}{q'}} < \infty,$$

where $P_Q(g)$ is determined by Lemma 6.13.

Lemma 6.15. ([5, 21]) $(H^p)^*(\mathbb{R}^n) = L(\frac{1}{p}, q', s)(\mathbb{R}^n)$, *where* $0 < p < 1 \leq q \leq \infty$, $s \in \mathbb{Z}$, $s \geq n(\frac{1}{p} - 1)$ *and* $1/q + 1/q' = 1$.

Now let us define a new Campanato-Meyers space with chirps as follows:

Definition 6.16. *For $s \in \mathbb{Z}^+$, $0 \le [n\beta] \le s$ and $1 \le q' \le \infty$. The chirp Campanato-Meyers space $L_\alpha(\beta, q', s)(\mathbb{R}^n)$ is defined as the set of locally integrable functions g that satisfy*

$$\|g\|_{L_\alpha(\beta,q',s)} = \sup_{Q\subset\mathbb{R}^n} |Q|^{-\beta} \left[\int_Q |e_\alpha(\boldsymbol{x})g(\boldsymbol{x}) - P_Q(e_\alpha g)(\boldsymbol{x})|^{q'} \frac{d\boldsymbol{x}}{|Q|} \right]^{\frac{1}{q'}} < \infty,$$

where $P_Q(e_\alpha g)$ is determined by Lemma 6.13.

Theorem 6.17. $(H^p_\alpha)^*(\mathbb{R}^n) = L_{-\alpha}(\frac{1}{p}, q', s)(\mathbb{R}^n)$*, where* $0 < p < 1 \le q \le \infty$, $s \in \mathbb{Z}$, $s \ge n(\frac{1}{p} - 1)$ *and* $1/q + 1/q' = 1$.

Proof. As in Theorem 6.11, the proof of this theorem can be obtained using Lemma 6.15; we omit the details. □

Remark 6.4. When $0 < p < 1$ and $\boldsymbol{\alpha} = (\frac{\pi}{2} + k_1\pi, \frac{\pi}{2} + k_2\pi, \ldots, \frac{\pi}{2} + k_n\pi), k_j \in \mathbb{Z}$ for $j = 1, 2, \ldots, m$, $(H^p_\alpha)^*(\mathbb{R}^n) = L_{-\alpha}(\frac{1}{p}, q', s)(\mathbb{R}^n)$ reduces to $(H^p)^*(\mathbb{R}^n) = L(\frac{1}{p}, q', s)(\mathbb{R}^n)$.

6.3. Characterization of the boundedness of singular integral operators with chirp functions on chirp Hardy spaces

In this subsection we obtain a characterization of the boundedness of T_α in H^p_α.

Theorem 6.18. *T_α is bounded from $H^p_\alpha(\mathbb{R}^n)$ to $H^p_\alpha(\mathbb{R}^n)$ if and only if T is bounded from $H^p(\mathbb{R}^n)$ to $H^p(\mathbb{R}^n)$, where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n$ with $\alpha_k \notin \pi\mathbb{Z}$ for all $k = 1, 2, \ldots, n$.*

Proof. Suppose that $f \in S'$ and $\|T_\alpha\|_{H^p_\alpha \to H^p_\alpha} < \infty$. Then, we have

$$\begin{aligned}
\|T(f)\|_{H^p} &= \left\| \sup_{t>0} |P_t * (e_\alpha(e_{-\alpha}Tf))| \right\|_{L^p} \\
&= \|T_\alpha(e_{-\alpha}f)\|_{H^p_\alpha} \\
&\le C\|e_{-\alpha}f\|_{H^p_\alpha} \\
&= C\|f\|_{H^p}.
\end{aligned}$$

Conversely, when $\|T\|_{H^p \to H^p} < \infty$, we obtain

$$\|T_\alpha(f)\|_{H^p_\alpha} = \left\| \sup_{t>0} |P_t * (T(e_\alpha f))| \right\|_{L^p} = \|T(e_\alpha f)\|_{H^p} \le C\|e_\alpha f\|_{H^p} = C\|f\|_{H^p_\alpha}.$$

Hence, the claim follows. □

Corollary 6.19. *Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n$ with $\alpha_k \notin \pi\mathbb{Z}$ for all $k = 1, 2, \ldots, n$. Then the fractional Riesz transform R^α_j is bounded from $H^p_\alpha(\mathbb{R}^n)$ to $H^p_\alpha(\mathbb{R}^n)$ if and only if the Riesz transform R_j is bounded from $H^p(\mathbb{R}^n)$ to $H^p(\mathbb{R}^n)$.*

7. Conclusions

In this paper, we introduced the fractional Riesz transforms and and studied certain a priori estimates of them in partial differential equations. We also studied properties of chirp singular integral operators, chirp Hardy spaces and chirp BMO spaces. We used the fractional Riesz transforms in concrete applications in edge detection with surprisingly good results. Our experiments indicate that edge detection can extract local information in any direction by adjusting the order of the fractional Riesz transform. The algorithm complexity of the fractional Riesz transform in (5.3) and (5.4) is reduced compared to that of the fractional Riesz transform of (5.1) and (5.2).

References

[1] A. Bultheel, H. Martínez-Sulbaran, Computation of the fractional Fourier transform, Appl. Comput. Harmon. Anal. 16 (2004), 182–202.

[2] A. Bultheel, H. Martínez-Sulbaran, `frft22d`: the matlab file of a 2D fractional Fourier transform, 2004. https://nalag.cs.kuleuven.be/research/software/FRFT/frft22d.m.

[3] A. P. Calderón, A. Zygmund, On singular integrals, Amer. J. Math. 78 (1956), 289–309.

[4] R. R. Coifman, A real variable characterization of H^p, Studia Math. 51 (1974), 269–274.

[5] R. R. Coifman, G. Weiss, Extensions of Hardy spaces and their use in analysis, Bull. Amer. Math. Soc. 83 (1977), 569–645.

[6] W. Chen, Z. W. Fu, L. Grafakos, Y. Wu, Fractional Fourier transforms on L^p and applications, Appl. Comput. Harmon. Anal. 55 (2021), 71–96.

[7] J. Duoandikoetxea, *Fourier analysis*, *Graduate Studies in Mathematics*, vol. 29, American Mathematical Society, Providence, RI, 2001.

[8] C. Fefferman, Characterizations of bounded mean oscillation, Bull. Amer. Math. Soc. 77 (1971), 587–588.

[9] C. Fefferman, E. M. Stein, H^p spaces of several variables, Acta Math. 129 (1972), 137–193.

[10] L. Grafakos, Classical Fourier analysis, 3rd ed., Graduate Texts in Mathematics, vol. 249, Springer, New York, 2014.

[11] L. Grafakos, Modern Fourier analysis, 3rd ed., Graduate Texts in Mathematics, vol. 250, Springer, New York, 2014.

[12] D. Gabor, Theory of communications, Inst. Elec. Eng, 93 (1946), 429–457.

[13] S. L. Hahn, *Hilbert transforms in signal processing*, Artech House Publish, 1996.

[14] F. H. Kerr, Namias fractional Fourier transforms on L^2 and applications to differential equations, J. Math. Anal. Appl. 136 (1988), 404–418.

[15] R. Kamalakkannan, R. Roopkumar, Multidimensional fractional Fourier transform and generalized fractional convolution, Integral Transforms Spec. Funct. 31 (2020), 152–165.

[16] K. Langley, S. J. Anderson, The Riesz transform and simultaneous representations of phaseenergy and orientation in spatial vision, Vision Research, 50 (2010), 1748–1765.
[17] S. Z, Lu. *Four lectures on real H^p spaces*, World Scientific Publishing Co. Inc., River Edge, NJ, 1995.
[18] S. Z, Lu, Y. Ding, D. Y. Yan, *Singular integrals and related topics,* World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2007.
[19] A. C. McBride, F. H. Kerr, On Namias's fractional Fourier transforms, IMA J. Appl. Math. 39 (1987), 159–175.
[20] V. Namias, The fractional order Fourier transform and its application to quantum mechanics, IMA J. Appl. Math. 25 (1980), 241–265.
[21] T. Walsh, The dual of $H^p(\mathbb{R}^{n+1}_+)$ for $p < 1$, Canad. J. Math. 25 (1973), 567–577.
[22] S. C. Pei, H. Yeh, Discrete fractional Hilbert transform, IEEE Trans Circuits and Systems-II. 47 (2000), 1307–1311.
[23] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions,* Princeton Mathematical Series, vol. 30. Princeton University Press, 1970.
[24] E. M. Stein, R. Shakarchi, *Functional analysis. Introduction to further topics in analysis,* Princeton Lectures in Analysis, 4. Princeton University Press, Princeton, NJ, 2011.
[25] M. H. Taibleson, G. Weiss, The molecular characterization of certain Hardy spaces, Asterisque, 1980.
[26] X. G. Xu, G. L. Xu, X. T. Wang, X. J. Qin, J. G. Wang, C. T. Yi, Review of bidimensional hilbert transform. Communications Technology. 49 (2016) 1265–1270.
[27] R. Tao, G. Liang, X. Zhao, An efficient FPGA-based implementation of fractional Fourier transform algorithm, J. Signal Processing Systems. 60 (2010) 47–58.
[28] R. Tao, X. M. Li, Y. Wang, Generalization of the fractional Hilbert transform, IEEE Signal Processing Letters. 15 (2008), 365–368.
[29] N. Wiener, Hermitian polynomials and Fourier analysis, J. Math. Phys. 8 (1929), 70–73.
[30] A. I. Zayed, Hilbert transform associated with the fractional Fourier transform, IEEE Signal Process. Lett. 5 (1998), 206–208.
[31] A. I. Zayed, Two-dimensional fractional Fourier transform and some of its properties, Integral Transforms Spec Funct. 29 (2018), 553–570.
[32] A. I. Zayed, A new perspective on the two-dimensional fractional Fourier transform and its relationship with the Wigner distribution, J. Fourier Anal. Appl. 25 (2019), 460–487.
[33] L. Zhang, H. Y. Li, Encoding local image patterns using Riesz transforms: With applications to palmprint and finger-knuckle-print recognition, Image and Vision Computing. 30 (2012) 1043–1051.
[34] L. Zhang, L. Zhang, X. Mou. RFSIM: A feature based image quality assessment metric using Riesz transforms. IEEE International Conference on Image Process-

ing. IEEE, 2010.